Strictly Lipschitz (p, r, s)-summing operators

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In this talk we present a new class of operators which is called strictly Lipschitz (p, r, s)-summing operators, knowing that the linear and the Lipschitz cases were given respectively by Lapreste and Chavez-Domìnguez, presenting also Dominition theorem and several properties.

In the same direction of ideas, we study the strong version of Lipschitz (p, r, s)-summing linear operators. The linear class has been stated by Lapreste in [3] and generalized to the Lipschitz case by Chávez-Domínguez [1]. Now, we recall the following definition as stated in [1]. Let X be a pointed metric space and E be a Banach space.

Now, we recall briefly some basic notations and terminology which we need in the sequel. Throughout this work, the letters E, F will denote Banach spaces and X, Y will denote metric spaces with a distinguished point (pointed metric spaces) which we denote by 0. Let X be a pointed metric space, we denote by $X^{\#} (= Lip_0(X))$ the Banach space of all Lipschitz functions $f : X \longrightarrow \mathbb{R}$ which vanish at 0 under the Lipschitz norm given by

$$Lip(f) = \sup\left\{\frac{|f(x) - f(y)|}{d(x, y)} : x, y \in X,\ x \neq y\right\}.$$

We denote by $\mathcal{F}(X)$, the Lipschitz-free Banach space over X, the closed linear span of the linear forms $\delta_{(x,y)}$ of $Lip_0(X)^*$ such that

$$\delta_{(x,y)}(f) = f(x) - f(y),\ \text{for every } f \in Lip_0(X),$$

i.e.,

$$\mathcal{F}(X) = \overline{span\left\{\delta_{(x,y)} : x, y \in X\right\}}^{Lip_0(X)^*}.$$

We have $X^{\#} = \mathcal{F}(X)^*$ holds isometrically via the application

$$Q_X(f)(m) = m(f),\ \text{for every } f \in X^{\#} \text{ and } m \in \mathcal{F}(X).$$

Let X be a pointed metric space and E be a Banach space, we denote by $Lip_0(X; E)$ the Banach space of all Lipschitz functions (Lipschitz operators) $T : X \to E$ such that $T(0) = 0$ with pointwise addition and Lipschitz norm. Note that for any $T \in Lip_0(X; E)$, then there exists a unique linear map (linearization of T) $\widehat{T} : \mathcal{F}(X) \longrightarrow E$ such that $\widehat{T} \circ \delta_X = T$ and $\left\|\widehat{T}\right\| = Lip(T)$, where δ_X is the canonical embedding so that $\langle \delta_X(x), f\rangle = \delta_{(x,0)}(f) = f(x)$ for $f \in X^{\#}$.

We denote by $l_p^n(E)$ the Banach space of all sequences $(e_i)_{i=1}^n$ in E with the norm

$$\|(e_i)_i\|_{l_p^n(E)} = (\sum_{i=1}^n \|e_i\|^p)^{\frac{1}{p}},$$

and by $l_p^{n,w}(E)$ the Banach space of all sequences $(e_i)_{i=1}^n$ in E with the norm

$$\|(e_i)_i\|_{l_p^{n,w}(E)} = \sup_{e^* \in B_{E^*}} (\sum_{i=1}^n |\langle e_i, e^*\rangle|^p)^{\frac{1}{p}}.$$

Definition 1. Let $0 < p, r, s < \infty$ with $\frac{1}{p} \leq \frac{1}{r} + \frac{1}{s}$, X be a pointed metric space and E be a Banach space. Let $T : X \to E$ be a Lipschitz map. T is Lipschitz (p, r, s)-summing if there is a constant $C > 0$ such that for any $n \in \mathbb{N}^*$, $(x_i)_i, (y_i)_i$ in X, $(e_i^*)_i$ in E^* and $(\lambda_i)_i, (k_i)_i$ in $\mathbb{R}_+^*$ $(1 \leq i \leq n)$, we have

$$\|(\lambda_i \langle T(x_i) - T(y_i), e_i^* \rangle)_i\|_{l_p^n} \leq C w_r^{Lip}\left(\left(\lambda_i k_i^{-1}, x_i, y_i\right)_i\right) \|(k_i e_i^*)_i\|_{l_s^{n,w}(E^*)}, \tag{1}$$

where $w_r^{Lip}\left(\left(\lambda_i k_i^{-1}, x_i, y_i\right)_{i=1}^n\right)$ is the weak Lipschitz p-norm defined by

$$\begin{aligned} w_r^{Lip}\left((\lambda_i, x_i, y_i)_{i=1}^n\right) &= \sup_{f \in B_{X^\#}} \left(\sum_{i=1}^n |\lambda_i (f(x_i) - f(y_i))|^r\right)^{\frac{1}{r}} \\ &= \left\|\left(\lambda_i \delta_{(x_i, y_i)}\right)\right\|_{l_r^{n,w}(\mathcal{F}(X))}. \end{aligned}$$

We denote by $\Pi_{p,r,s}^L(X, E)$ the Banach space of all Lipschitz (p, r, s)-summing operators with the norm $\pi_{p,r,s}^L(T)$ which is the smallest constant C such that inequality (1) holds.

Definition 2. Let $0 < p, r, s < \infty$ such that $\frac{1}{p} \leq \frac{1}{r} + \frac{1}{s}$. Let X be a pointed metric space and E be a Banach space. The Lipschitz operator $T : X \to E$ is strictly Lipschitz (p, r, s)-summing if there is a constant $C > 0$ such that for any $n_1 \in \mathbb{N}^*, n_2 \in \mathbb{N}^*$, $\left(x_i^j\right)_{i=1}^{n_1}, \left(y_i^j\right)_{i=1}^{n_1} \subset X$, $\left(\lambda_i^j\right)_{i=1}^{n_1} \subset \mathbb{K}$ $(j = 1, ..., n_2)$ and any $e_1^*, ..., e_{n_1}^* \in E^*$, we have

$$\left(\sum_{i=1}^{n_1} \left|\left\langle \sum_{j=1}^{n_2} \lambda_i^j (T(x_i^j) - T(y_i^j)), e_i^* \right\rangle\right|^p\right)^{\frac{1}{p}} \leq C \sup_{f \in B_{X^\#}} \left(\sum_{i=1}^{n_1} \left|\sum_{j=1}^{n_2} \lambda_i^j (f(x_i^j) - f(y_i^j))\right|^r\right)^{\frac{1}{r}} \|(e_i^*)\|_{l_s^{n_1 w}(E^*)}. \tag{2}$$

The class of all strictly Lipschitz (p, r, s)-summing operators from X into E is denoted by $\Pi_{p,r,s}^{SL}(X, E)$, which is a Banach space with the norm $\pi_{p,r,s}^{SL}(T)$ which is the smallest constant C such that inequality (2) holds.

Theorem 1. *Let $0 < p, r, s < \infty$ with $\frac{1}{p} = \frac{1}{r} + \frac{1}{s}$. Let X be a pointed metric space and E be a Banach space. Let $T : X \to E$ be a Lipschitz operator. The following properties are equivalent*

1. *T is strictly Lipschitz (p, r, s)-summing.*
2. *$\widehat{T}$ is (p, r, s)-summing. In this case we have*

$$\pi_{p,r,s}\left(\widehat{T}\right) = \pi_{p,r,s}^{SL}(T).$$

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