Darboux transformation of the generalized Jacobi matrices and Toda lattice

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We study a Darboux transformation of the symmetric Jacobi matrix J associated with the Toda lattice. We find factorization of the Jacobi matrix J, i.e $J = \mathfrak{U}\mathfrak{L}$, where $\mathfrak{L}$ and $\mathfrak{U}$ are the lower and upper triangular two-diagonal matrices, respectively. Darboux transformation of J was found, one is the symmetric Jacobi matrix $J^{(d)} = \mathfrak{L}\mathfrak{U}$, which is associated with the another Toda lattice.

Definition 1. The Toda lattice is a system of differential equations

$$x_n''(t) = e^{x_{n-1}-x_n} - e^{x_n - x_{n+1}}, \quad n \in \mathbb{N}, \tag{1}$$

which was introduced in [1].

We investigate the semi-infinite system with $x_1 = -\infty$. The Flaschka variables are defined by

$$a_k^2 = e^{x_{k-1}-x_k} \quad \text{and} \quad b_k = -x_k' \qquad (2).$$

Therefore, we obtain the following system in terms of the Flaschka variables

$$(a_k^2)' = a_k^2(b_k - b_{k-1}) \quad \text{and} \quad b_k' = 2(a_{k+1}^2 - a_k^2), \quad a_0 = 0 \qquad (3).$$

On the other hand, the symmetric Jacobi matrix J associated with the Toda lattice, which is defined by

$$J = \begin{pmatrix} b_0 & a_1 & & \\ a_1 & b_1 & a_2 & \\ & a_2 & b_2 & \ddots \\ & & \ddots & \ddots \end{pmatrix}.$$

Theorem 1. *Let J be the symmetric Jacobi matrix and let S_0 be a some real parameter. Then J admits the following $\mathfrak{U}\mathfrak{L}$–factorization*

$$J = \mathfrak{U}\mathfrak{L},$$

where $\mathfrak{L}$ and $\mathfrak{U}$ are the lower and upper triangular matrices, respectively, which are defined by

$$\mathfrak{L} = \begin{pmatrix} 1 & & & \\ \dfrac{S_0+b_0}{a_1} & 1 & & \\ & \dfrac{S_1+b_1}{a_2} & 1 & \\ & & \ddots & \ddots \end{pmatrix} \quad \text{and} \quad \mathfrak{U} = \begin{pmatrix} -S_0 & a_1 & & \\ & -S_1 & a_2 & \\ & & -S_2 & \ddots \\ & & & \ddots \end{pmatrix},$$

if and only if the following system is solvable

$$S_i(S_{i-1} + b_{i-1}) = -a_i^2, \quad S_{i-1} + b_{i-1} \neq 0 \quad \text{and} \quad S_{i-1} \neq 0, \text{ for all } i \in \mathbb{N}.$$

Definition 2. Let the symmetric Jacobi matrix J admit $\mathfrak{UL}$ – factorization. Then a transformation

$$J = \mathfrak{UL} \to \mathfrak{LU} = J^{(d)}$$

is called a Darboux transformation with parameter of the Jacobi matrix J.

Theorem 2. *Let the symmetric Jacobi matrix J admit $\mathfrak{UL}$ with parameter $S_0 \in \mathbb{R}\backslash\{0, -b_0\}$. Then the Darboux transformation with parameter of the Jacobi matrix J is the symmetric Jacobi matrix*

$$J^{(d)} = \begin{pmatrix} -S_0 & a_1 & & \\ a_1 & b_0 & a_2 & \\ & a_2 & b_1 & \ddots \\ & & \ddots & \ddots \end{pmatrix}$$

if and only if

$$S_0 = S_i \quad \text{for all } i \in \mathbb{N}.$$

Theorem 3. *Let the symmetric Jacobi matrix J admit $\mathfrak{UL}$–factorization and J be associated with the Toda lattice* (1)–(3)*. Let the symmetric Jacobi matrix $J^{(d)} = \mathfrak{LU}$ be the Darboux transformation without parameter of J. Then $J^{(d)}$ is associated with the following Toda lattice*

$$x_k''(t) = e^{x_{k-1}-x_k} - e^{x_k - x_{k+1}},$$

$$a_k^2 = e^{x_{k-1}-x_k}, \quad S_0 = x_0' \quad \text{and} \quad b_{k-1} = -x_k'.$$

$$a_0 = 0, \quad (a_1^2)' = a_1^2(b_0 + S_0), \quad (a_k^2)' = a_k^2(b_{k+1} - b_k),$$
$$-S_0' = 2(a_1^2 - a_0^2) \quad \text{and} \quad b_{k-1}' = 2(a_{k+1}^2 - a_k^2), \quad k \in \mathbb{N}.$$

1. Toda M. Theory of Nonlinear Lattices. Springer Series in Solid-State Sciences, **20** (2 ed.), Springer, Berlin, 1989.