Fixed point theorem for non self operators on Strictly Star Shaped Sets of Generalized Banach Spaces and applications

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In this talk, we are going to give a generalization of Leray-Schauder fixed point theorem in Generalized Banach spaces for non-self set contractive mappings defined in non-convex domains via the vector valued measure of noncompactness tool. Let us begin with the definition of generalized Banach space.

Definition 1. Let $\mathcal{E}$ be a vector space on $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. By a generalized norm on $\mathcal{E}$ we mean a map

$$\|\cdot\|_G : \mathcal{E} \longrightarrow \mathbb{R}^n_+$$
$$\varrho \mapsto \|\varrho\|_G = \big(\|\varrho\|_1, \cdots, \|\varrho\|_n\big)^n$$

satisfying the following properties:

(i) For all $\varrho \in \mathcal{E}$; if $\|\varrho\|_G = 0_{\mathbb{R}^n_+}$, then $\varrho = 0$,

(ii) $\|\lambda\varrho\|_G = |\lambda|\|\varrho\|_G$ for all $\varrho \in \mathcal{E}$ and $\lambda \in \mathbb{K}$, and

(iii) $\|\varrho + y\|_G \preccurlyeq \|\varrho\|_G + \|y\|_G$ for all $\varrho, y \in \mathcal{E}$.

The pair $(\mathcal{E}, \|\cdot\|_G)$ is called a vector (generalized) normed space. Furthermore, $(\mathcal{E}, \|\cdot\|_G)$ is called a generalized Banach space (in short, GBS), if the vector metric space generated by its vector metric is complete.

Definition 2. Let $(\mathcal{E}, \|\cdot\|_G)$ be a GBS and let $\mathcal{B}_G(\mathcal{E})$ be the family of G-bounded subsets of $\mathcal{E}$. A map

$$\mu_G : \mathcal{B}_G(\mathcal{E}) \longrightarrow [0, +\infty)^n$$
$$\mathcal{A} \mapsto \mu_G(\mathcal{A}) = \big(\mu_1(\mathcal{A}), \cdots, \mu_n(\mathcal{A})\big)^T$$

is called a generalized measure of noncompactness (for short G-MNC) defined on $\mathcal{E}$ if it satisfies the following conditions:

(i) The family $\operatorname{Ker}\mu_G(\mathcal{E}) = \{\mathcal{A} \in \mathcal{B}_G(\mathcal{E}) : \mu_G(\mathcal{A}) = 0\}$ is nonempty and $\operatorname{Ker}\mu_G(\mathcal{E}) \subset \mathcal{N}_G(\mathcal{E})$.
(ii) Monotonicity: $\mathcal{A}_1 \subseteq \mathcal{A}_2 \Rightarrow \mu_G(\mathcal{A}_1) \preccurlyeq \mu_G(\mathcal{A}_2)$, for all $\mathcal{A}_1, \mathcal{A}_2 \in \mathcal{B}_G(\mathcal{E})$.
(iii) Invariance under closure and convex hull: $\mu_G(\mathcal{A}) = \mu_G(\overline{\mathcal{A}}) = \mu_G(co(\mathcal{A}))$, for all $\mathcal{A} \in \mathcal{B}_G(\mathcal{E})$.
(iv) Convexity: $\mu_G(\lambda\mathcal{A}_1 + (1-\lambda)\mathcal{A}_2) \preccurlyeq \lambda\mu_G(\mathcal{A}_1) + (1-\lambda)\mu_G(\mathcal{A}_2)$, for all $\mathcal{A}_1,\ \mathcal{A}_2 \in \mathcal{B}_G(\mathcal{E})$ and $\lambda \in [0,1]$.
(v) Generalized Cantor intersection property: If $(\mathcal{A}_m)_{m\geq 1}$ is a sequence of nonempty, closed subsets of $\mathcal{E}$ such that $\mathcal{A}_1$ is G-bounded and $\mathcal{A}_1 \supseteq \mathcal{A}_2 \supseteq \ldots \supseteq \mathcal{A}_m \ldots$, and $\lim_{m\to+\infty}\mu_G(\mathcal{A}_m) = 0_{\mathbb{R}^n_+}$, then the set $\mathcal{A}_\infty := \bigcap_{m=1}^{\infty}\mathcal{A}_m$ is nonempty and is G-compact.
Moreover, we say that μ_G is:
(vi) Semi-additive if $\mu_G(\mathcal{A}_1 \cup \mathcal{A}_2) = \max\{\mu_G(\mathcal{A}_1) + \mu_G(\mathcal{A}_2)\}$, for all $\mathcal{A}_1,\ \mathcal{A}_2 \in \mathcal{B}_G(\mathcal{E})$.

Definition 3 ([2])**.** Let $\mathcal{E}$ be a GBS and let $\mathcal{K} \subset \mathcal{E}$ be an open G-bounded, with $0 \in \mathcal{K}$. We say that $\mathcal{K}$ is strictly star-shaped with respect to 0 if for all $\varrho \in \partial\mathcal{K}$, $\{t\varrho : t > 0\} \cap \partial\mathcal{K} = \{\varrho\}$.

Definition 4. A matrix $M \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$ is said to be convergent to zero if $M^m \longrightarrow 0$, as $\quad m \longrightarrow \infty$.

Definition 5. Let $(\mathcal{E}, \|\cdot\|_G)$ be a GBS and let μ_G be a G-MNC. A self-mapping $N : \mathcal{E} \longrightarrow \mathcal{E}$ is said to be satisfies the generalized Darbo condition with respect to μ_G if N maps G-bounded operators, and there exists a matrix $M \in \mathcal{M}_{n\times n}(\mathbb{R}^+)$ converges to zero such that
$\mu_G(N(\mathcal{A})) \preccurlyeq M\mu_G(\mathcal{A})$, for every $\mathcal{A} \in \mathcal{B}_G(\mathcal{E})$.

Now, we state and prove our main results.

Theorem 1 ([1])**.** *Let $\mathcal{E}$ be a GBS, μ_G be a semi-additive G-MNC, $\mathcal{K} \subset \mathcal{E}$ be a G-bounded, strictly star-shaped open neighborhood of zero and let $N : \overline{\mathcal{K}} \to \mathcal{E}$ be a continuous operator satisfies the generalized Darbo condition with a matrix M. If N satisfies the Leray-Schauder boundary condition, then N has at least one fixed point.*

By using the above fixed point result, we prove the existence of solutions for the following coupled system of nonlinear integral operators :

$$\begin{cases} \varrho_1(\tau) = \displaystyle\int_0^\tau \kappa_1(\tau,\sigma) l_1(\tau,\sigma,\varrho_1(\sigma),\varrho_2(\sigma)) d\sigma, \\ \varrho_2(\tau) = \displaystyle\int_0^\tau \kappa_2(\tau,\sigma) l_2(\tau,\sigma,\varrho_1(\sigma),\varrho_2(\sigma)) d\sigma. \end{cases} \tag{1}$$

In GBS $\mathcal{E} = \mathcal{C}(J,\mathbb{R}) \times \mathcal{C}(J,\mathbb{R})$ of all couple of continuous functions on $J = [0,b]$, $0 < b < \infty$. where, the functions κ_1, κ_2, l_1, l_2 are given and verify the next assumptions

$(\mathcal{H}_1)$ κ_1, $\kappa_2 : J \times J \longrightarrow \mathbb{R}$ are continuous, and there exists constants $\xi_i \in \mathbb{R}_*^+$ such that

$$|\kappa_i(\tau,\sigma)| \le \xi_i \qquad \text{for all } (\tau,\sigma) \in J^2.$$

$(\mathcal{H}_2)$ The functions l_1, $l_2, : J \times J \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ such that

- (a) For an arbitrary fixed and $\varrho = (\varrho_1, \varrho_2) \in \mathbb{R} \times \mathbb{R}$, the function $(\tau,\sigma) \to l_i(\tau,\sigma,\varrho_1,\varrho_2)$, $i = 1,2$ is continuous on $\bar{J} \times \bar{J}$,
- (b) There exists a matrix $M = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathcal{M}_{2\times 2}(\mathbb{R}_+)$ such that for each $(\tau,\sigma,\varrho_1,\varrho_2), (\tau,\sigma,\bar{\varrho}_1,\bar{\varrho}_2) \in J \times J \times \mathbb{R} \times \mathbb{R}$ and for $i \in \{1,2\}$, we have

$$|l_i(\tau,\sigma,\varrho_1,\varrho_2) - l_i(\tau,\sigma,\bar{\varrho}_1,\bar{\varrho}_2)| \le a_{i1}|\varrho_1 - \bar{\varrho}_1| + a_{i2}|\varrho_2 - \bar{\varrho}_2|.$$

Theorem 2. *Suppose that the assumptions $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$ are satisfied. Assume that there is $R \in \mathbb{R}_+^2$ and u, $v \in B(0,R) \subset \mathcal{E}$, such that for each $i = 1,\ 2$ one of the following inequality holds*

$$\|u_i\|_\infty < R_i - \xi_i\alpha_i b, \qquad \|v_i\|_\infty < R_i - \xi_i\alpha_i b,$$

where $\alpha_i = (R_1 a_{i,1} + R_2 a_{i,2}) + \sup_{(\iota,\varsigma)\in J^2} |l_i(\iota,\varsigma,0,0)|$. Then, the SIE (1) has at least one solution.

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