(p, q)-PSEUDOSPECTRUM AND (p, q)-CONDITION SPECTRUM

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We introduce the (p, q)-pseudospectrum and (p, q)-condition spectrum of an element in Banach algebra, and show their properties for an element represented as a block matrix.

Let $\mathcal{A}$ be a complex Banach algebra with the unit 1. We use $\mathcal{A}^{-1}$, $\mathcal{A}^{\bullet}$, and $\sigma(a)$ to denote the set of all invertible and idempotent elements in $\mathcal{A}$, and the spectrum of a, respectively.

We can represent an element of Banach algebra in a block matrix form as follows. Let $u \in \mathcal{A}^{\bullet}$. Then we can represent an element $a \in \mathcal{A}$ as $a = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_u$, where $a_{11} = uau$, $a_{12} = ua(1-u)$, $a_{21} = (1-u)au$, $a_{22} = (1-u)a(1-u)$. Note that $u\mathcal{A}u$ and $(1-u)\mathcal{A}(1-u)$ are Banach algebras with the units u and $1-u$, respectively. In the following text, we will consider diagonal matrix representation

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}_u . \tag{1}$$

An element $a \in \mathcal{A}$ is outer generalized invertible, if there exists some $b \in \mathcal{A}$ satisfying $b = bab$. Such b is called the outer generalized inverse of a. In this case ba and $1 - ab$ are idempotents corresponding to a and b. The set of all outer generalized invertible elements of $\mathcal{A}$ will be denoted with $\mathcal{A}^{(2)}$.

The outer generalized inverses with prescribed idempotents were defined in [1] as follows: Let $a \in \mathcal{A}$ and $p, q \in \mathcal{A}^{\bullet}$. An element $b \in \mathcal{A}$ satisfying $bab = b, ba = p, 1 - ab = q$, will be called a (p, q)-outer generalized inverse of a, written $a^{(2)}_{p,q} = b$. If $a^{(2)}_{p,q}$ exists, then it is unique. The set of all outer generalized invertible elements of $\mathcal{A}$ with prescribed idempotents $p, q \in \mathcal{A}^{\bullet}$ will be denoted with $\mathcal{A}^{(2)}_{p,q}$.

The pseudospectrum and the condition spectrum were defined and studied in [3] and [4]:

• Pseudospectrum [4] : Let $\varepsilon > 0$. The ε-pseudospectrum of an element $a \in \mathcal{A}$ is defined as $\Lambda_\varepsilon(a) = \{z \in \mathbb{C} \mid a - z \notin \mathcal{A}^{-1} \text{ or } ||(a-z)^{-1}|| \geq \varepsilon^{-1}\}$.

• Condition spectrum [3]: Let $0 < \varepsilon < 1$. The ε-condition spectrum of an element $a \in \mathcal{A}$ is defined as $\sigma_\varepsilon(a) = \{z \in \mathbb{C} \mid a - z \notin \mathcal{A}^{-1} \text{ or } ||(a-z)^{-1}|| \cdot ||a - z|| \geq \varepsilon^{-1}\}$.

We generalize the pseudospectrum and the condition spectrum by defining the (p, q)-pseudospectrum and the (p, q)-condition spectrum as follows:

Definition 1. ((p, q)-pseudospectrum) Let $\varepsilon > 0$. The (p, q)-ε-pseudospectrum of an element $a \in \mathcal{A}$ is defined as $\Lambda_\varepsilon(a) = \left\{z \in \mathbb{C} \mid a - z \notin \mathcal{A}^{(2)}_{p,q} \text{ or } ||(a-z)^{(2)}_{p,q}|| \geq \varepsilon^{-1}\right\}$.

Definition 2. ((p, q)-condition spectrum) Let $0 < \varepsilon < 1$. The (p, q)-ε-condition spectrum of $a \in \mathcal{A}$ is defined as $\sigma_{(p,q)-\varepsilon}(a) = \left\{z \in \mathbb{C} \mid a - z \notin \mathcal{A}^{(2)}_{p,q} \text{ or } ||(a-z)^{(2)}_{p,q}|| \cdot ||a - z|| \geq \varepsilon^{-1}\right\}$.

Now, we state the auxiliary results.

Lemma 1. *Let x be as in (1), $p_1, q_1 \in (u\mathcal{A}u)^\bullet$ and $p_2, q_2 \in ((1-u)\mathcal{A}(1-u))^\bullet$ and let $p = p_1 + p_2 \in \mathcal{A}$ and $q = q_1 + q_2 \in \mathcal{A}$. Then $x \in \mathcal{A}^{(2)}_{p,q}$ if and only if $a \in (u\mathcal{A}u)^{(2)}_{p_1,q_1}$ and $b \in ((1-u)\mathcal{A}(1-u))^{(2)}_{p_2,q_2}$. If $x \in \mathcal{A}^{(2)}_{p,q}$, then $x^{(2)}_{p,q} = \begin{bmatrix} a^{(2)}_{p_1,q_1} & 0 \\ 0 & b^{(2)}_{p_2,q_2} \end{bmatrix}_u$.*

As a corollary, we have the result for the invertibility of an element represented as in (1).

Lemma 2. *Let x be as in (1). Then $x \in \mathcal{A}^{-1}$ if and only if $a \in (u\mathcal{A}u)^{-1}$ and $b \in ((1-u)\mathcal{A}(1-u))^{-1}$. If $x \in \mathcal{A}^{-1}$, then $x^{-1} = \begin{bmatrix} a^{-1} & 0 \\ 0 & b^{-1} \end{bmatrix}_u$.*

Therefore, it holds $\sigma(x) = \sigma(a) \cup \sigma(b)$ for the spectrum of an element x represented as in (1).

We investigate whether the similar property holds for the (p,q)-pseudospectrum and the (p,q)-condition spectrum. It turned out that this property depends on the norm used in Banach algebra for the elements represented as in (1).

Theorem 1. *Let x be as in (1) with the norm $||x||_\infty = \max\{||a||, ||b||\}$, $\varepsilon > 0$, $p_1, q_1 \in (u\mathcal{A}u)^\bullet$ and $p_2, q_2 \in ((1-u)\mathcal{A}(1-u))^\bullet$ and let $p = p_1 + p_2 \in \mathcal{A}$ and $q = q_1 + q_2 \in \mathcal{A}$. Then $\Lambda_{(p_1,q_1)-\varepsilon}(a) \cup \Lambda_{(p_2,q_2)-\varepsilon}(b) = \Lambda_{(p,q)-\varepsilon}(x)$.*

Theorem 2. *Let x be as in (1) with the norm $||x||_1 = ||a|| + ||b||$, $\varepsilon > 0$, $p_1, q_1 \in (u\mathcal{A}u)^\bullet$ and $p_2, q_2 \in ((1-u)\mathcal{A}(1-u))^\bullet$ and let $p = p_1 + p_2 \in \mathcal{A}$ and $q = q_1 + q_2 \in \mathcal{A}$. Then $\Lambda_{(p_1,q_1)-\varepsilon}(a) \cup \Lambda_{(p_2,q_2)-\varepsilon}(b) \subset \Lambda_{(p,q)-\varepsilon}(x)$.*

The following theorem holds whether we consider the norm $||\cdot||_\infty$ or the norm $||\cdot||_1$.

Theorem 3. *Let x be as in (1), $0 < \varepsilon < 1$, $p_1, q_1 \in (u\mathcal{A}u)^\bullet$ and $p_2, q_2 \in ((1-u)\mathcal{A}(1-u))^\bullet$ and let $p = p_1 + p_2 \in \mathcal{A}$ and $q = q_1 + q_2 \in \mathcal{A}$. Then $\sigma_{(p_1,q_1)-\varepsilon}(a) \cup \sigma_{(p_2,q_2)-\varepsilon}(b) \subset \sigma_{(p,q)-\varepsilon}(x)$.*

The next example shows that the converse inclusion is not true in the previous theorem.

Example 1. Let $0 < \varepsilon < 1$, $z \in \mathbb{C}$ and $u \in \mathcal{A}^\bullet$ such that $||u|| < \frac{1}{\sqrt{\varepsilon}}$ and $||1-u|| < \frac{1}{\sqrt{\varepsilon}}$. Let $x = \begin{bmatrix} (\varepsilon^2 + z)u & 0 \\ 0 & (\varepsilon + z)(1-u) \end{bmatrix}_u \in \mathcal{A}$ relative to the idempotent $u \in \mathcal{A}$. Then $z \in \sigma_{(1,0)-\varepsilon}(x)$, but $z \notin (\sigma_{(u,0)-\varepsilon}((\varepsilon^2 + z)u) \cup \sigma_{(1-u,0)-\varepsilon}((\varepsilon + z)(1-u)))$.

The new result is Theorem 2, while the other results presented here were published in the author's previous paper [2].

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