On the closedness of the sum of subspaces of the space $B(H,Y)$ consisting of operators whose kernels contain given subspaces of H

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1. Let X be a real or complex Banach space. By a subspace of X we will mean a linear subset of X (thus a subspace of X *is not* necessarily closed in X). Let n be a natural number, $n \geqslant 2$, and let $X_1, \dots, X_n$ be subspaces of X. Define their sum in the natural way, namely,

$$X_1 + \ldots + X_n := \Big\{ x_1 + \ldots + x_n \,|\, x_1 \in X_1, \ldots, x_n \in X_n \Big\}.$$

It is clear that $X_1 + \ldots + X_n$ is a subspace of X. Assume that $X_1, \ldots, X_n$ are closed in X. The natural question arises: is $X_1 + \ldots + X_n$ closed in X? The question makes sense — the sum of two closed subspaces can be non-closed. In fact, in every infinite-dimensional Banach space X there exist two closed subspaces X_1 and X_2 such that $X_1 + X_2$ is not closed in X, see [3, Proposition in subsection 4.8 on p. 122]. A simple example of non-closedness (more precisely, a class of examples) can be obtained as follows (this example is the straightforward generalization of the example of two closed subspaces of a Hilbert space with non-closed sum given in [2, p. 110]). Let V and W be Banach spaces over the same field of real or complex scalars and $A\colon V \to W$ be a continuous linear operator with non-closed range $Ran(A)$. Set $X := V \oplus W$ with the norm $\|(v, w)\| = \|v\| + \|w\|$, $v \in V$, $w \in W$, and let $X_1 = V \oplus \{0\} = \{(v, 0) \,|\, v \in V\}$, $X_2 = Graph(A) = \{(v, Av) \,|\, v \in V\}$. It is clear that X_1 and X_2 are closed in X but their sum $X_1 + X_2 = V \oplus Ran(A)$ is not.

Systems of closed subspaces of Banach spaces for which the closedness of their sum is important arise in various branches of mathematics, see the list of such branches in subsection 1.4 of [1].

2. Let X and Y be Banach spaces over the same field of real or complex numbers. Denote by $B(X, Y)$ the linear space of all continuous linear operators $A\colon X \to Y$ endowed with the standard operator norm, i.e., $\|A\| = \sup\{\|Ax\|/\|x\| \,|\, x \in X, x \neq 0\}$, $A \in B(X, Y)$. Then $B(X, Y)$ is a Banach space. Let X_0 be a closed subspace of X. Denote by $Z(X_0; X, Y)$ the set of all operators $A \in B(X, Y)$ such that $Ax = 0$ for every $x \in X_0$. (The letter "Z" in $Z(X_0; X, Y)$ comes from the word "zero".) In other words, $Z(X_0; X, Y)$ is the set of all operators $A \in B(X, Y)$ such that $X_0 \subset Ker(A)$. It is clear that $Z(X_0; X, Y)$ is a closed subspace of $B(X, Y)$.

3. We study the following questions. Let n be a natural number, $n \geqslant 2$, and let $X_1, \ldots, X_n$ be closed subspaces of X.

Question 1 *(on the closedness)***:** when is $Z(X_1; X, Y) + \ldots + Z(X_n; X, Y)$ closed in $B(X, Y)$?

Before formulating the second question, we note that always

$$Z(X_1; X, Y) + \ldots + Z(X_n; X, Y) \subset Z(X_1 \cap X_2 \cap \ldots \cap X_n; X, Y).$$

Question 2 *(on the maximality)***:** when is $Z(X_1; X, Y) + \ldots + Z(X_n; X, Y)$ equal to $Z(X_1 \cap X_2 \cap \ldots \cap X_n; X, Y)$?

Of course, Question 2 is related to Question 1 — if $Z(X_1; X, Y) + \ldots + Z(X_n; X, Y) = Z(X_1 \cap X_2 \cap \ldots \cap X_n; X, Y)$, then $Z(X_1; X, Y) + \ldots + Z(X_n; X, Y)$ is closed in $B(X, Y)$.

4. We will answer Questions 1 and 2 in the case when $X = H$ is a Hilbert space. To formulate our result we need the following notation. Let H be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$. Two elements x_1 and x_2 of the space H are said to be orthogonal if $\langle x_1, x_2 \rangle = 0$. For a closed subspace H_0 of H we denote by $H_0^\perp$ the orthogonal complement of H_0 in H, i.e., the set of all elements of the space H which are orthogonal to every element of the subspace H_0. In other words, $H_0^\perp$ is the set of all $x \in H$ such that $\langle x, x_0 \rangle = 0$ for every $x_0 \in H_0$. It is clear that $H_0^\perp$ is a closed subspace of H.

Now we are ready to formulate our result.

Theorem 1. *Let H be a Hilbert space and Y be a Banach space over the same field of real or complex numbers. Let n be a natural number, $n \geqslant 2$, and $H_1, \ldots, H_n$ be closed subspaces of H. The following statements are equivalent:*

a) *the subspace $Z(H_1; H, Y) + \ldots + Z(H_n; H, Y)$ is closed in $B(H, Y)$;*

b) $Z(H_1; H, Y) + \ldots + Z(H_n; H, Y) = Z(H_1 \cap H_2 \cap \ldots \cap H_n; H, Y)$;

c) *the subspace $H_1^\perp + \ldots + H_n^\perp$ is closed in H.*

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