Sequences in Banach spaces

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Recall that a continuous linear operator T between Banach spaces is absolutely p-summing (p-summing for short) if it sends weakly p-summable sequences to absolutely p-summable sequences. Karn and Sinha [2] introduce the mid-p-summable (operator p-summable) sequences in a Banach space E such that $\ell_p(E) \subset \ell_p^{mid}(E) \subset \ell_p^w(E)$, they introduce the weakly mid-p-summing (sequentially p-limited) operators. A bounded linear operator $T : E \to F$ between Banach spaces is weakly mid-p-summing if it sends weakly p-summable sequences to mid-p-summable sequences. Equivalently $S \circ T : E \to \ell_p$ is p-summing for all bounded linear operator S from F to ℓ_p. The principal idea is to replace, in the definition of p-summing operators, p-summable and weakly p-summable sequences by Lorentz mid-(p,q)-summable and weakly Lorentz (p,q)-summable [1] sequences respectively. Equivalently, ℓ_p and p-summing operators space by $\ell_{p,q}$ and Lorentz (p,q)-summing operators. In this work, we introduce and study the Lorentz mid-(p,q)-summable sequence in a Banach space E, denoted $\ell_{p,q}[E]$ as a generalization of the mid-p-summable sequence space. We prove that $\ell_{p,q}(E) \subset \ell_{p,q}[E] \subset \ell_{p,q}^w(E)$, and we give some properties concerning this Banach sequences space, we prove that $\ell_{p,q}[E] = \ell_{p,q}^w(E)$ if and only if $\Pi_{p,q}^{lz}(E, \ell_{p,q}) = \mathcal{L}(E, \ell_{p,q})$.

Definition 1. Let $1 \leq q \leq p < \infty$. We say that $(x_j)_{j=1}^\infty$ in Banach space E is a Lorentz mid-(p,q)-summable sequence if $((\langle x_n^*, x_j\rangle)_{j=1}^\infty)_{n=1}^\infty \in \ell_{p,q}(\ell_{p,q})$ for any $(x_n^*)_{n=1}^\infty \in \ell_{p,q}^{w^*}(E^*)$. The space of all such sequences shall be denoted by $\ell_{p,q}[E]$.

Definition 1 is also equivalent to: A sequence $(x_j)_{j=1}^\infty$ in Banach space E is Lorentz mid-(p,q)-summable if $((\langle x_n^*, x_j\rangle)_{n=1}^\infty)_{j=1}^\infty \in \ell_{p,q}(\ell_{p,q})$ for any $(x_n^*)_{n=1}^\infty \in \ell_{p,q}^{w^*}(E^*)$.

Proposition 1. *The next expression*

$$\|(x_j)_{j=1}^\infty\|_{[p,q]} := \sup_{(x_n^*)_{n=1}^\infty \in B_{\ell_{p,q}^{w^*}(E^*)}} \|((\langle x_n^*, x_j\rangle)_{j=1}^\infty)_{n=1}^\infty\|_{\ell_{p,q}(\ell_{p,q})}$$

is a norm that makes $\ell_{p,q}[E]$ a Banach space.

Since $\ell_{p,q}^{w^*}(E^*)$ can be identified with $\mathcal{L}(E, \ell_{p,q})$ and $\ell_{p,q}^{w^*}(E^*) = \ell_{p,q}^w(E^*)$, the above definition 1 can be reorganized as follows.

Proposition 2. *$(x_j)_{j=1}^\infty \in \ell_{p,q}[E]$ if and only if $(S(x_j))_{j=1}^\infty \in \ell_{p,q}(\ell_{p,q})$ for all $S \in \mathcal{L}(E, \ell_{p,q})$. Moreover,*

$$\|(x_j)_{j=1}^\infty\|_{[p,q]} = \sup\{\|(S(x_j))_{j=1}^\infty\|_{\ell_{p,q}(\ell_{p,q})} : S \in \mathcal{L}(E, \ell_{p,q}),\ \|S\| \leq 1\}.$$

Proposition 3. *Given $1 \leq q \leq p < \infty$. We have*

$$\ell_{p,q}(E) \subset \ell_{p,q}[E] \subset \ell_{p,q}^w(E),$$

moreover

$$\|(x_j)_j\|_{p,q}^w \leq \|(x_j)_j\|_{[p,q]} \leq \|(x_j)_j\|_{p,q}.$$

We say that a Banach space E is a Lorentz weak mid-(p,q)-space if $\ell_{p,q}[E] = \ell^w_{p,q}(E)$.

Theorem 1. *Let E be a Banach space. Then, E is Lorentz weak mid-(p,q)-space if and only if $\mathcal{L}(E, \ell_{p,q}) = \Pi^{lz}_{p,q}(E, \ell_{p,q})$.*

1. M. C. Matos and D. Pellegrino. Lorentz summing mappings. Math Nachr, 2010, 283, 10, 1409–1427 .
2. A. Karn and D. Sinha. An operator summability of sequences in Banach spaces. Glasg. Math. J, 2014, 56, 2, 427–437.