Explicit rank-one constructions for irrational rotations

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By a dynamical system we mean a quadruple $(X, \mathfrak{B}, \mu, T)$, where $(X, \mathfrak{B})$ is a standard Borel space, μ is a σ-finite measure on $\mathfrak{B}$ and T is an invertible μ-preserving transformation of X.

Definition 1. A dynamical system $(X, \mathfrak{B}, \mu, T)$ (or just T) is called

- *ergodic* if each T-invariant subset is either μ-null or μ-conull;
- *totally ergodic* if T^p is ergodic for each $p \in \mathbb{N}$;
- *of rank one* if there are subsets $B_1, B_2, \dots$ of X with $\mu(B_k) < \infty$ for each $k \in \mathbb{N}$ and a sequence of positive integers $n_1 < n_2 < \cdots$ such that $T^l B_k \cap T^m B_k = \emptyset$ whenever $0 \le l < m < n_k$ and $k \in \mathbb{N}$ and for each subset $A \in \mathfrak{B}$ of finite measure,
$$\lim_{k\to\infty} \min_{J\subset\{0,\dots,n_k-1\}} \mu\left(A \triangle \bigsqcup_{j\in J} T^j B_k\right) = 0;$$
- *of* 0*-type* if
$$\mu(T^n A \cap B) \to 0 \quad \text{as } n \to \infty$$
for all subsets $A, B \in \mathfrak{B}$ with $\mu(A) < \infty$ and $\mu(B) < \infty$.

There is an alternative (explicit) definition of rank one system via an inductive construction process of cutting-and-stacking with a single tower on each step. It is completely determined by two underlying sequences of *cuts* and *spacer mappings*.

Let $\theta \in (0, 1)$ be an irrational number and let $\lambda := e^{2\pi i\theta}$. Denote by R_λ the λ-rotation on the circle $\mathbb{T}$. Del Junco showed in [1] that R_λ is of rank one if $\mathbb{T}$ is furnished with the Haar measure. He also raised a related (more subtle) problem in [2]:

Problem 1. Given θ, provide an *explicit* construction (i.e. find sequences of cuts and spacer mappings) of a rank-one transformation which is isomorphic to R_λ.

A solution of Problem 1 for an uncountable subset of well approximable irrationals of zero Lebesgue measure was found recently in [3].

We consider also a weak version of Problem 1.

Problem 2. Given θ, provide an *explicit* construction of a rank-one probability preserving transformation T which has an eigenvalue λ.

Del Junco solved Problem 2 for a.e. $\theta \in (0, 1)$ in [2].

We now state the main results of the first part (related to the probability preserving systems) of the present abstract.

Main Result A.

- *Problem* 2 *is solved for every* θ.
- *Problem* 1 *is solved for each well approximable* θ.
- *For almost all* $\theta \in (0,1)$, *including the badly approximable reals and the algebraic numbers, we solve Problem* 2 *in the subclass of rank-one transformations with only two cuts at every step of their inductive construction.*

As the subset of well approximable reals from $(0,1)$ is of Lebesgue measure 1, Problem 1 is solved "almost surely".

In the second part of the paper we consider Problem 1 within the class of the infinite measure preserving dynamical systems. The main difference from the probability preserving case is that for each irrational θ, there exist *uncountably many* mutually disjoint R_λ-invariant infinite σ-finite measures on $\mathbb{T}$ (see, e.g., [4]). The main result of the second part is contained in the following theorem.

Theorem 1. *For each element* $\lambda \in \mathbb{T}$ *of infinite order, there is an infinite* σ*-finite* R_λ*-invariant non-atomic Borel measure* μ'_λ *on* $\mathbb{T}$ *such that*

- *the dynamical system* $(\mathbb{T}, \mu'_\lambda, R_\lambda)$ *is of rank one, the parameters of the underlying cutting-and-stacking construction are explicitly described,*
- $(\mathbb{T}, \mu'_\lambda, R_\lambda)$ *is totally ergodic and of zero type,*
- $C(R_\lambda) = \{R_\lambda^n \mid n \in \mathbb{Z}\}$,
- $\mu'_\lambda \circ R_\beta \perp \mu'_\lambda$ *whenever* $\beta \notin \{\lambda^n \mid n \in \mathbb{Z}\}$ *and*
- *if an element* $\omega \in \mathbb{T}$ *is of infinite order with* $\omega \notin \{\lambda, \lambda^{-1}\}$ *then* $\mu'_\omega \perp \mu'_\lambda$.

As far as we know, Theorem 1 provides the first examples of *spectrally mixing*[1] ergodic infinite invariant measures for irrational rotations.

Everywhere in the paper we construct the rank-one systems via the (C,F)-construction. It was introduced in [4] and [5] (in a different form). Various kinds of the (C,F)-construction, interrelationship among them and the classical cutting-and-stacking are discussed in detail in [6].

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[1] An infinite measure preserving transformation S is of zero type if and only if the measure of maximal spectral type of S is Rajchman, i.e. the Koopman operator associated with S is mixing.