Sparse approximation of compact break-of-scale embeddings

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This talk is concerned with the approximation of functions belonging to mixed smoothness Besov spaces $S^r_{p,q}B([0,1]^d)$ or Sobolev spaces $S^r_pH([0,1]^d)$. It represents the results obtained in [1]. Our interest is measuring approximation errors in the energy norm $H^1([0,1]^d)$ or more general in isotropic Sobolev spaces $H^s_q([0,1]^d)$. The interest for this setting is motivated by worst-case error analysis of Galerkin discretizations methods for PDEs. The quantities of interest are best approximation from a linear Daubechies wavelet subspace

$$d_m(I;D) := d_m(I\colon S^{r_0}_{p_0,q_0}B \to H^{s_1}_{p_1};D) := \inf_{\substack{\Lambda_m\subset\Lambda,\\ |\Lambda_m|\le m}} \sup_{\|f|S^{r_0}_{p_0,q_0}B\|\le 1} \inf_{\substack{c_\lambda\in\mathbb{C},\\ \lambda\in\Lambda_m}} \left\| If - \sum_{\lambda\in\Lambda_m} c_\lambda\, b^\lambda \;\middle|\; H^{s_1}_{p_1}\right\|,$$

and best m-term approximation given by

$$\sigma_m(I\colon S^{r_0}_{p_0,q_0}B \to H^{s_1}_{p_1};D) := \sup_{\|f|S^{r_0,s_0}_{p_0,q_0}B\|\le 1} \inf_{\substack{\Lambda_m\subset\Lambda,\\ |\Lambda_m|\le m}} \inf_{\substack{c_\lambda\in\mathbb{C},\\ \lambda\in\Lambda_m}} \left\| If - \sum_{\lambda\in\Lambda_m} c_\lambda\, b^\lambda \;\middle|\; H^{s_1}_{p_1}\right\|,$$

where $\mathcal{D}$ is the dictionary of sufficient smooth hyperbolic Daubechies wavelets. In [2] it was proven that isotropic Sobolev spaces can be equivalently characterized by hyperbolic Daubechies wavelets that are typically used for mixed smoothness spaces. The result are comparable discretization spaces on the one hand for isotropic and on the other hand for mixed smoothness spaces. This allows us to transfer our approximation interest to discrete sequence spaces. Our main result reads as follows

Theorem 1. *Let $0 < p_0 \le \infty$, and $1 < p_1 < \infty$ as well as $r, s \in \mathbb{R}$ such that*

$$r - \left(\frac{1}{p_0} - \frac{1}{p_1}\right)_+ > s > 0. \tag{1}$$

1. *If $1 < p_0 < \infty$, then the embedding $Id_1\colon S^r_{p_0}H([0,1]^d) \to H^s_{p_1}([0,1]^d)$ is compact and there holds*

$$d_m(Id_1;\Psi) \sim m^{-[r-s-(1/p_0-1/p_1)_+]} \qquad \textit{as well as} \qquad \sigma_m(Id_1;\Psi) \sim m^{-(r-s)}.$$

2. *If further $0 < q_0 \le \infty$, then the embedding $Id_2\colon S^r_{p_0,q_0}B([0,1]^d) \to H^s_{p_1}([0,1]^d)$ is compact, where*

$$d_m(Id_2;\Psi) \sim m^{-[r-s-(1/p_0-1/p_1)_+]} \qquad \textit{and} \qquad \sigma_m(Id_2;\Psi) \sim m^{-(r-s)}.$$

We observe that the asymptotic approximation rates do not depend on the dimension d of the underlying domain. Additionally we see that the best m-term approximation in this break-of-scale situation does not reflect the integrability parameter in the approximation rate.

1. Byrenheid Glenn, Huebner Janina, Weimar Markus. Rate-optimal sparse approximation of compact break-of-scale embeddings. Applied and Computational Harmonic Analysis, 2023, 65, 40–66.
2. Schaefer Martin, Ullrich Tino, Vedel Beatrice. Hyperbolic wavelet analysis of classical isotropic and anisotropic Besov-Sobolev spaces. (English summary) J. Fourier Anal. Appl., 2021.