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On Symmetric Polyconvexity of Compact Sets

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Quasiconvexity was firstly introduced by C. B. Morrey in 1952, see [3], but it is usually difficult to check if a given function is quasiconvex. Two other notions were introduced, one necessary: rank-one-convexity and the other sufficient: polyconvexity. A function is polyconvex if it can be written as a convex expression of the minors.

Definition 1. A function f is rank-one-convex if the function $f(F+tE)$ is a convex function for every matrix F and any matrix E of rank one.

The relation between the different notions of convexity is given by the following implications

$$(f, \text{convex}) \underset{\not\Leftarrow}{\Longrightarrow} (f, \text{polyconvex}) \underset{\not\Leftarrow}{\Longrightarrow} (f, \text{quasiconvex}) \underset{\Longleftarrow\,\mathbf{?}}{\overset{\Longrightarrow}{2\times 2}} (f, \text{rank-one-convex}).$$

In the 2×2 case, whether rank-one-convexity is equivalent to quasiconvexity or not is still an open problem, see [2] for more details and results.

In the context of non linear elasticity, the set $K = W^{-1}\{0\}$ is a zero-set of the energy. The smallest set containing K and all affine deformation with zero-energy is the set K^{qc}, which is the quasiconvex hull of the set K. The big challenge is how to determine K^{qc} for a given set K.

Recall that the classical convex hull K^c is characterized by

$$K^c = \{\sum_1^{N+1} a_i F_i : a_i \in (0,1), F_i \in K, \sum a_i = 1\}.$$

Such characterization for quasiconvex hull K^{qc} is missing. Nevertheless

$$K^{qc} = \{F : f(F) \leq \sup_K f, \text{ for any quasiconvex function } f\}.$$

Remark 1.

- The quasiconvex hull of a set K is the set of points which can not be separated from K by quasiconvex functions.
- In the same way we define respectively the rank-one-convex hull K^{rc} and the polyconvex hull K^{rc} of the set K by replacing quasiconvexity by rank-one-convexity and polyconvexity respectively.
- Due to the relations between the different notions of convexity, we have

$$K \subset K^{rc} \subset K^{qc} \subset K^{pc} \subset K^c.$$

The set of convex combination of rank one connected elements of K is in general different from K, we denote it K^{lc}, and is known as Lamination convex hull

$$K^{lc} = \cup_{i \geq 0} K^i,$$

where $K^0 = K$ and by induction

$$K^{i+1} = \{\lambda A + (1-\lambda)B,\, \lambda \in (0,1),\ A, B \in K^i\},$$

$$K \subset K^{lc} \subset K^{rc} \subset K^{qc} \subset K^{pc} \subset K^c.$$

A set K is said to be quasicovex (resp. rank-one-convex, polyconvex, lamination convex) if it is equal to it's quasiconvex (resp. rank-one-convex, polyconvex, lamination convex) hull K^{qc}, (resp. K^{rc}, K^{pc}, K^{lc}).

In the symmetric case we consider compact sets, $K \subset S^{d\times d}$. We are interested in computing of the symmetric quasiconvex hull of the set K. The symmetric semi convex hull for symmetric sets are defined in the same way as above, we replace convexity (quasiconvexity, rank-one-convexity) by symmetric convexity (symmetric quasiconvexity, symmetric rank-one-convexity), for more details on the notions of symmetric semiconvexity, the reader is referred to [1].

The symmetric lamination convex hull of K is defined as lamination convex hull by replacing rank one connection by the compatibility condition. Two symmetric matrices are compatibles if the difference is the symmetric part of a rank one matrices.

$$A, B \text{ compatible if and only if } A - B = \frac{1}{2}(a \otimes b)(b \otimes a).$$

We define by the same way the notion of symmetric convexity (sc) (quasiconvexity (sqc) and rank-one-convexity(src)). For the quasiconvexity we just take the symmetric part of $\nabla\psi$ instead of $\nabla\psi$. In this case also one have

$$K \subset K^{slc} \subset K^{src} \subset K^{sqc} \subset K^{sc}.$$

Few people know about the computation of symmetric semi convex hull, even in the simplest case. If K is a set of three incompatible symmetric matrices in $S^{3\times 3}$, then $K^{slc} = K$ but K^{sqc} and even K^{src} is still unknown. Our contribution is the following

Theorem 1. *We consider any compact set $K \subset \mathbb{R}^{2\times 2}$, then K is symmetric polyconvex if and only if for any $\lambda_i \in [0,1]$, $i = 1, 2, \dots, 5$ and any family of matrices $\epsilon_i \in K$, $i = 1, 2, \dots, 5$ such that $\det \sum_{i=1}^{5} \lambda_i \epsilon_i \geq \sum_{i=1}^{5} \lambda_i \det \epsilon_i$ holds $\sum_{i=1}^{5} \lambda_i \epsilon_i \in K$.*

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