Boas type theorems for the one-dimensional $(k;a)$-generalized Fourier transform

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We define and study properties related to the (k,a)-generalized translation operator which is induced by the product formula of two kernels $B_{k,a}(\lambda,x)B_{k,a}(\lambda,y)$ (see, e.g., [1]) associated with the $(k;a)$-generalized Fourier transform. We study Boas type theorems associated with the $(k;a)$-generalized Fourier transform, more precisely; we define the modulus of smoothness associated with the $(k;a)$-generalized Fourier transform. Moreover, we obtain B-N-S inequality for this transform. As application, we prove a connection between K-functionals and a modulus of smoothness in $L^2(\mu_{k,a})$. We also study Jackson's theorem associated with the $(k;a)$-generalized Fourier transform.

Lemma 1. *Let $k > \frac{3-2a}{4}$ and $|\lambda x| \geq 1$. Then we have the following assertions*

(i) $|1 - B_{k,a}(\lambda,x)| \geq A$,
where A is a positive constant.

(ii) The kernel $B_{k,a}(\lambda,.)$ has the asymptotic behavior in 0

$$B_{\lambda}^{k,n}(x) = 1 - \frac{4|\lambda x|^a}{8ka+4a^2-4a} + m_{k,a}\lambda x + sgn(\lambda x) m_{k,a}\frac{4|\lambda x|^{a+1}}{8ka+4a^2+4a} + o(|\lambda x|^{a+1}).$$

Lemma 2. *(See, e.g., [2]) The $(k;a)$-generalized Fourier transform $B_{k,a}(\lambda,.)$ possesses the following properties.*

(i) $\mathcal{L}_{k,a}B_{k,a}(\lambda,x) = -|\lambda|^a B_{k,a}(\lambda,x)$.

(ii) $|B_{k,a}(\lambda,x)| \leq 1$,

where $\mathcal{L}_{k,a}$ is the $(k;a)$-generalized Fourier oscillator (see, e.g., [2]).

A natural application of the product formula of two (k,a)-generalized Fourier kernel is to define the translation operator.

Definition 1. (see, e.g., [1]) Let $x > 0$ and f a continuous bounded function. Then, the (k,a)-generalized Fourier translation operator τ_y is defined by

$$\tau_y f(x) = \int_{\mathbb{R}} f(z)d\gamma_{x,y}(z), \quad x,y \in \mathbb{R}, \tag{1}$$

where

$$d\gamma_{x,y}(z) = \begin{cases} \Delta_{k,a}(x,y,z)|z|^{2k+a-2}dz, & \text{if } xy \neq 0; \\ \delta_x(z), & \text{if } y = 0; \\ \delta_y(z), & \text{if } x = 0. \end{cases}$$

The translation operator (1) satisfies the following:

Proposition 1. *Let $\alpha < \frac{1}{2}$, for all $x, y \in \mathbb{R}_+$ and f is a continuous bounded function. Then*

(i) $\tau_x f(y) = \tau_y f(x)$, (ii) $\tau_0 f(x) = f(x)$, (iii) $\tau_x\tau_y = \tau_y\tau_x$, (iv) $\|\tau_x f\|_{k,a,p} \leq 4\|f\|_{k,a,p}$, (v) If we suppose also that $f \in \mathcal{C}_c(\mathbb{R})$, then $\mathcal{L}_{k,a}\tau_x = \tau_x\mathcal{L}_{k,a}$, and (vi) $\mathcal{F}_{k,a}(\tau_y f)(\lambda) = B_{k,a}(\lambda, (-1)^{\frac{2}{a}} y)\mathcal{F}_{k,a}(f)(\lambda)$.

Definition 2. We define the mth-order finite differences operator with step h by

$$\Delta_h^m f(x) = \sum_{l=0}^{m} (-1)^l \binom{m}{l} \tau_{lh} f(\lambda).$$

Lemma 3. *Let $f \in L^2(\mu_{k,a})$. Then $\|\Delta_h^m f\|_{k,a,2} \leq 5^m \|f\|_{k,a,2}$.*

Proposition 2. *For all $f \in L^p(\mu_{k,a})$, we have*

$$\mathcal{F}_{k,a}(\Delta_h^m f)(\lambda) = \tilde{B}^{k,a,m}(\lambda h)\mathcal{F}_{k,a} f(\lambda), \quad \textit{where} \quad \tilde{B}^{k,a,m}(x) = \sum_{l=0}^{m} (-1)^l \binom{m}{l} B_{k,a}(l, x).$$

Lemma 4. *For all function $f \in \mathcal{W}_2^{k,a,m}, t > 0$. The following inequality holds:*

$$\omega_{2,m}^{k,a}(f, \delta) \leq c_1 t^m \|\mathcal{L}_{k,a}^m f\|_{k,a,2}.$$

Lemma 5. *For all function $f \in L^2(\mu_\alpha)$ the following inequality holds:*

$$\|f - P_\beta(f)\|_{k,a,2} \leq c_2 \|\Delta_{1/\beta}^m f\|_{k,a,2}, \quad \beta > 0,$$

where P_β is defined for all $f \in L^2(\mu_{k,a})$ by $P_\beta(f)(x) = F_W^{-1}(F_W(f)(x)\mathbb{1}_\beta(x)), \quad \beta > 0.$

Theorem 1. *For all function $f \in L^2(\mu_{k,a})$ the following inequality holds:*

$$\|L^m(P_\beta(f))\|_{k,a,2} \leq c_4 \beta^m \|\Delta_{1/\beta}^m f\|_{k,a,2}, \quad \beta > 0, \quad m \in \mathbb{N}.$$

Theorem 2. *For all positive constants $c_1 = c_1(m, k, a)$ and $c_2 = c_2(m, k, a)$. Then the following inequality holds*
$c_1\omega_{2,m}^{k,a}(f, \delta) \leq K_{2,m}^{k,a}(f, \delta^m) \leq c_2\omega_{2,m}^{k,a}(f, \delta)$, *where $f \in L^2(\mu_{k,a})$, $\delta > 0$.*

Theorem 3. *Suppose that $f \in \mathcal{W}_2^{k,a,m}$ $(m = 1, 2, ...)$. Then for all $\beta > 0$, we have*

$$E_\beta^{k,a,2}(f) \leq c\beta^{-2m}\omega_2^{k,a}(\mathcal{L}_{k,a}^m f, 1/\beta),$$

where $E_\beta^{k,a,2}$ is the value of the best approximation of a function $f \in L^2(\mu_{k,a})$ and $\omega_2^{k,a}(\delta, f) = \|\tau_\delta f - f\|_{k,a,2}$ the modulus of continuity of f.

Note that, the investigated results given in this abstract are analogs of those studied by (see, e.g., [3, 5]) and jointly with Dr. Ahmed Saoudi (see, e.g., [4]).

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