The Bergman Popov's subgradient extragradient algorithm for strongly pseudomonotone equilibrium problems

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Consider the equilibrium problem $EP(f;\ C)$ as follows

$$\text{find } x \in C \ \text{ such that } \quad f(x,y) \geq 0,\, \forall y \in C,$$

where C is a nonempty, closed and convex subset of a real linear space E, $f : C \times C \to \mathbb{R}$ be a bifunction. We denote by E^* the dual of E. Muu and Oettli are first one introduced the term of the equilibrium problem in 1992 ([2]) and has been extended by Blum and Oettli ([1]). The problems $EP(f;\ C)$ have a number of interesting explanations and are related to many branches of pure and applied mathematics : variational inequality and fixed point problems, nonlinear optimization. The problems $EP(f;\ C)$ is also considered as a generalization of the convex minimization problems. The special case, if $f(x,y) =< Ax, y-x >$, where $A : E \to E^*$, the equilibrium problem is just the variational inequality problem $VI(A;C)$ which gives by

$$\text{find } x \in C \ \text{ such that } < Ax, y-x > \geq 0,\, \forall y \in C.$$

Let $h : E \to \mathbb{R}$ be a convex differentiable. The Bregman distance is the bifunction

$$D_h : dom(f) \times int(dom(h)) \longrightarrow [0,+\infty)\,,$$

which is defined by

$$D_h(x,y) := h(x) - h(y) - < \nabla h(y), x-y > .$$

Here $h : E \to \mathbb{R}$ is differentiable, continuous and strongly convex with constant $\sigma > 0$, i.e.,

$$h(x) - h(y) \geq \langle \nabla h(y), y-x \rangle + \frac{\sigma}{2} \left\| x-y \right\|^2 .$$

In general, the Bregman distance is not symmetric and the triangle inequality does not hold. However, it is also considered a generalization of some well-known distances. It is also called the three point identity: for any $x \in dom(f)$ and $y,\ z \in int(dom(f))$

$$D_h(x,y) + D_h(y,z) - D_h(x,z) =< \nabla h(z) - \nabla h(y), x-y > .$$

From the strong convexity of h, we have

$$D_h(x,y) \geq \frac{\sigma}{2} \left\| x-y \right\|^2,\ \forall x,y \in C.$$

In this talk, by using Bergman distance, we introduce iterative extension of the Popov's subgradient extragradient method for solving strongly pseudomonotone equilibrium problems. Let the bifunction $f : E \times E \to \mathbb{R}$ satisfies the following conditions

(C1) $f(x,x) = 0$, for all $x \in C$ and f is strongly pseudomonotone on C, i.e.,

$$f(x,y) \geq 0 \Rightarrow f(y,x) \leq -\gamma \left\| x-y \right\|^2, \qquad \forall x,y \in C$$

(C2) f is Bregman-Lipschitz-type condition, i.e., there exist two positive constants c_1, c_2, such that

$$f(x,y) + f(y,z) \geq f(x,z) - c_1 D_h(y,x) - c_2 D_h(z,y), \qquad \forall x,y,z \in C.$$

(C3) $f(x,\cdot)$ is convex and subdifferentiable on E for each fixed $x \in E$. Using the concept of Bregman distance, we introduce our method.

Algorithm 1. *(The Bergman Popov's subgradient extragradient method for SPEP) Choose $x_0, y_0 \in H$, and a sequence $\{\lambda_n\}$ satisfying the following conditions*

(Cd1): $\lim\limits_{n\to+\infty} \lambda_n = 0$ *and* *(Cd2):* $\sum\limits_{n=1}^{+\infty} \lambda_n = +\infty.$

Set

$$\begin{cases} x_1 = \arg\min\limits_{y\in C} \{\lambda_0 f(y_0,y) + D_h(y,x_0)\} \\ y_1 = \arg\min\limits_{y\in C} \{\lambda_1 f(y_0,y) + D_h(y,y_0)\} \end{cases}$$

***Iterative steps**: Given x_n, y_{n-1}, and y_n for $n \geq 1$. Construct a half-space*

$$H_n = \{z \in H : \langle \nabla h(x_n) - \lambda_n v_{n-1} - \nabla h(y_n), z - y_n \rangle \leq 0\},$$

where $v_{n-1} \in \partial_2 f(y_{n-1}, y_n)$.

***Step 1**: Compute*

$$x_{n+1} = \arg\min_{y\in H_n} \{\lambda_n f(y_n,y) + D_h(y,x_n)\},$$

***Step 2**: Compute*

$$y_{n+1} = \arg\min_{y\in C} \{\lambda_{n+1} f(y_n,y) + D_h(y,x_{n+1})\}.$$

*If $x_{n+1} = x_n = y_n$, then we stop. Otherwise, set $n := n+1$ and go to the **Iterative steps**.*

Lemma 1. *For all $p^* \in EP(f,\mathbb{C})$, the following inequality holds*

$$D_h(p^*, x_{n+1}) \leq D_h(p^*, w_n) - (1+\lambda_n c_2) D_h(z_n, y_n) - D_h(y_n, w_n) - c_1 \lambda_n D_h(y_n, y_{n-1}) - \gamma \lambda_n \|y_n - p^*\|^2.$$

Theorem 1. *Let $f : H \times H \to \mathbb{R}$ be a bifunction satisfying the conditions (C1)–(C2). Then the sequences $\{x_n\}$ and $\{y_n\}$, generated by Algorithm 1, are converges strongly to $p^* \in EP(f,\mathbb{C})$. Moreover, $\lim\limits_{n\to+\infty} P_{EP(f,\mathbb{C})}(x_n) = p^*$.*

1. Blum E., Oettli W. From optimization and variational inequality to equilibrium problems. Math. Stud, 1994, 63, 127–149.
2. Muu L. D., Oettli W. Convergence of an adaptive penalty scheme for finding constrained equilibria, Nonlinear Anal, 1992, 18, 1159–1166.