Norm inequalities for positive semedifinite matrices

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Let $M_n(\mathbb{C})$ denote the space of all $n \times n$ complex matrices. The singular values of a matrix $A \in M_n(\mathbb{C})$, denoted by $s_1(A) \geq ... \geq s_n(A)$, are the eigenvalues of the positive semidefinite matrix $|A| = (A^*A)^{1/2}$ repeated according to multiplicity. A matrix norm $|||\cdot|||$ on $M_n(\mathbb{C})$ is called unitarily invariant if $|||UAV||| = |||A|||$ for all $A \in M_n(\mathbb{C})$ and for all unitary matrices $U, V \in M_n(\mathbb{C})$. The Schatten p-norm of $A \in M_n(\mathbb{C})$ is defined by

$$||A||_p = \left(\sum_{i=1}^{n} s_i^p(A) \right)^{1/p}$$

for $1 \leq p \leq \infty$. If $p = \infty$, we get the spectral norm, which is denoted by $||A|| = s_1(A)$.

For $A, B \in M_n(\mathbb{C})$, the direct sum $A \oplus B$ is $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$.

As basic properties of the direct sum, we have $\|A \oplus B\| = \max(\|A\|, \|B\|)$ and $\|A \oplus B\|_p = \left(\|A\|_p^p + \|B\|_p^p \right)^{1/p}$. Also $\left|\left|\left| \begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix} \right|\right|\right| = |||A \oplus A^*||| = |||A \oplus A|||$.

In this talk, we prove several unitarily invariant norm inequalities for positive semidefinite matrices. Some of these results give generalizations of earlier known inequalities.

Theorem 1. *Let $A, B, X, Y \in M_n(\mathbb{C})$ be such that A and B are positive semidefinite. Then*

$$\begin{aligned} |||XAY - YBX||| & \\ \leq\ & \|X\|\|Y\|\, |||A \oplus B||| \\ & + \frac{1}{2} \left|\left|\left| \left(A^{1/2}(X^*Y - YX^*)B^{1/2} \right) \oplus \left(B^{1/2}(Y^*X - XY^*)A^{1/2} \right) \right|\right|\right| \end{aligned}$$

for every unitarily invariant norm. In particular, if $X^*Y = YX^*$, then

$$|||XAY - YBX||| \leq \|X\|\|Y\|\, |||A \oplus B|||\,.$$

Theorem 2. *Let $A, B, X, Y \in M_n(\mathbb{C})$ be such that A and B are positive semidefinite. Then*

$$\begin{aligned} \left|\left|\left| A^{1/4}(X^*Y - YX^*)B^{1/4} \right|\right|\right| & \\ \leq\ & \|X\|\|Y\| \left|\left|\left| A^{1/2} \oplus B^{1/2} \right|\right|\right| \\ & + \frac{1}{2} \left|\left|\left| \left(A^{1/4}(X^*Y - YX^*)B^{1/4} \right) \oplus \left(B^{1/4}(Y^*X - XY^*)A^{1/4} \right) \right|\right|\right| \end{aligned}$$

for every unitarily invariant norm.

Theorem 3. *Let $A, B, X, Y \in M_n(\mathbb{C})$ be such that A and B are positive semidefinite. Then*

$$s_j(A^{1/2}XB^{1/2} + B^{1/2}YA^{1/2}) \leq \|A+B\| s_j(X \oplus Y)$$

for $j = 1, ..., n$.

Theorem 4. *Let $A, B, X, Y \in M_n(\mathbb{C})$ be such that A and B are positive semidefinite matrices, and let $f(t)$ be a nonnegative concave function on $[0, \infty)$ satisfying $f(0) = 0$. Then*

$$\left|\left|\left| f\left(\left|A^{1/2}XB^{1/2} + B^{1/2}YA^{1/2}\right|\right)\right|\right|\right| \leq \left|\left|\left| f\left(\frac{B + X^*AX}{2}\right)\right|\right|\right| + \left|\left|\left| f\left(\frac{B + YAY^*}{2}\right)\right|\right|\right|$$

for every unitarily invariant norm.

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