JAFARI TRANSFORM OF MABC FRACTIONAL INTEGRAL OPERATOR

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The Jafari integral transform of the integrable function $v(t)$ denoted by $\mathcal{F}(\psi(s),\phi(s))$ is recently introduced by Hossein Jafari [4] by:

$$\mathcal{J}[v(t)] = \mathcal{F}(\psi(s),\phi(s)) = \phi(s)\int_0^\infty v(t)\exp^{(-\psi(s)t)}\,dt,$$

where $t \geq 0$ and $\phi(s), \psi(s) : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that $\phi(s) \neq 0$.

Definition 1. [2,3] For $v^{(n)} \in L^1(0,\infty)$, $n-1<\sigma<n$, $n \in \mathbb{N}$, $\alpha = \sigma - n + 1$, $0<\alpha<1$, the higher order Atangana–Baleanu fractional derivative of Caputo sense is defined by

$$\left({}^{ABC}\mathbf{D}_0^\sigma v\right)(t) = \frac{M(\alpha)}{1-\alpha}\int_0^t E_\alpha\left(-r_\alpha (t-s)^\alpha\right) v^{(n)}(s)\,ds, \quad t \geq 0,$$

where $M(\alpha)$ is a normalization function satisfying $M(0)=M(1)=1$ and $r_\alpha = \frac{\alpha}{1-\alpha}$.

Definition 2. [1] For $v \in L^1(0,\infty)$, $0<\alpha<1$, the modified Atangana–Baleanu fractional integral operator is defined by

$$\left({}^{ABM}I_0^\alpha v\right)(t) = \frac{1-\alpha}{M(\alpha)}v(t) + \frac{\alpha}{M(\alpha)}\left({}^{RL}I_0^\alpha v\right)(t) - \frac{1-\alpha}{M(\alpha)}v(0)\left(1 + \frac{r_\alpha}{\Gamma(1+\alpha)}t^\alpha\right), \quad (1)$$

where ${}^{RL}I_0^\alpha$ is the well known Riemann-Liouville fractional integral.

Moreover, it holds that

$$\left({}^{ABM}I_0^\alpha v\right)(0) = \frac{\alpha}{M(\alpha)}\left({}^{RL}I_0^\alpha v\right)(0).$$

The modified integral operator in (1) can be written as

$$\left({}^{ABM}I_0^\alpha v\right)(t) = \frac{1-\alpha}{M(\alpha)}(v(t)-v(0)) + \frac{\alpha}{M(\alpha)}\left(\left({}^{RL}I_0^\alpha(v-v(0))\right)\right)(t).$$

When $\alpha = 0$ we recover the initial function, and if $\alpha = 1$, we obtain the ordinary integral.

Definition 3. [1] For $v \in L^1(0,\infty)$, $n-1<\sigma<n$, $n \in \mathbb{N}$, $\alpha = \sigma - n + 1$, the modified higher order Atangana–Baleanu fractional integral operator is defined by

$$\begin{aligned}\left({}^{ABM}\mathbf{I}_0^\sigma v\right)(t) &= \frac{1-\alpha}{M(\alpha)}\left[\left({}^{RL}I_0^{n-1}v\right)(t) + r_\alpha\left({}^{RL}I_0^{n+\alpha-1}v\right)(t) - v(0)\left(\frac{t^{n-1}}{\Gamma(n)} + r_\alpha\frac{t^{n+\alpha-1}}{\Gamma(n+\alpha)}\right)\right],\\ &= \frac{1-\alpha}{M(\alpha)}\left[\left({}^{RL}I_0^{n-1}v - v(0)\right)(t) + r_\alpha\left({}^{RL}I_0^{n+\alpha-1}v - v(0)\right)(t)\right]. \quad (2)\end{aligned}$$

Lemma 1. *[1] For $v^{(n)} \in L^1(0,\infty)$, $n-1<\sigma<n$, $n\in\mathbb{N}$, $\alpha=\sigma-n+1$, then*

$$\left({}^{ABM}\boldsymbol{I}_0^{\sigma}\, {}^{ABC}\boldsymbol{D}_0^{\sigma} v\right)(t) = v(t) - \sum_{n-1}^{k=0} v^{(k)}(0)\frac{t^k}{k!};$$

$$\left({}^{ABC}\boldsymbol{D}_0^{\sigma}\, {}^{ABM}\boldsymbol{I}_0^{\sigma} v\right)(t) = v(t) - v(0).$$

In the following, let $v(t)\in A$ with new general transform $\mathsf{F}(s)$.

Lemma 2. *The new general transform of the modified Atangana–Baleanu fractional integral operator (1) is*

$$\mathcal{J}\left[\left({}^{ABM}I_0^{\alpha} v\right)(t)\right] = \frac{1-\alpha}{M(\alpha)}\left(1+\frac{r_\alpha}{\psi(s)^\alpha}\right)\left[\mathcal{F}(\psi(s),\phi(s)) - \frac{\phi(s)}{\psi(s)}v(0)\right].$$

Remark 1. If $\phi(s)=1$ and $\psi(s)=s$, then the Laplace transform is given by [1], direct calculation will lead to

$$\mathcal{J}\left[\left({}^{ABM}I_0^{\alpha} v\right)(t)\right] = \mathcal{L}\left[\left({}^{ABM}I_0^{\alpha} v\right)(t)\right] = \frac{1-\alpha}{M(\alpha)}\left(1+\frac{r_\alpha}{s^\alpha}\right)\left[\mathcal{F}(\psi(s),\phi(s)) - \frac{1}{s}v(0)\right],$$

Remark 2. If $v(0)=0$, then $\left({}^{ABM}I_0^{\alpha} v\right)(t) = \left({}^{AB}I_0^{\alpha} v\right)(t)$, consequently we get

$$\mathcal{J}\left[\left({}^{ABM}I_0^{\alpha} v\right)(t)\right] = \mathcal{J}\left[\left({}^{AB}I_0^{\alpha} v\right)(t)\right] = \frac{1-\alpha}{M(\alpha)}\left(1+\frac{r_\alpha}{\psi(s)^\alpha}\right)\mathcal{F}(\psi(s),\phi(s)).$$

Lemma 3. *The new general transform of the modified higher order Atangana–Baleanu fractional integral operator is defined by (2) is*

$$\mathcal{J}\left[\left({}^{ABM}\boldsymbol{I}_0^{\sigma} v\right)(t)\right] = \frac{1-\alpha}{M(\alpha)}\frac{\psi(s)^\alpha + r_\alpha}{\psi(s)^{n+\alpha-1}}\left[\mathcal{F}(\psi(s),\phi(s)) - \frac{\phi(s)}{\psi(s)}v(0)\right].$$

Remark 3. If $\phi(s)=1$ and $\psi(s)=s$, we get

$$\mathcal{J}\left[\left({}^{ABM}\mathbf{I}_0^{\sigma} v\right)(t)\right] = \mathcal{L}\left[\left({}^{ABM}\mathbf{I}_0^{\sigma} v\right)(t)\right] = \frac{1-\alpha}{M(\alpha)}\frac{s^\alpha + r_\alpha}{s^{n+\alpha-1}}\left[\mathcal{F}(\psi(s),\phi(s)) - \frac{1}{s}v(0)\right],$$

and this is the same result obtained in [1].

1. Al-Refai M. Proper inverse operators of fractional derivatives with nonsingular kernels. Rend. Circ. Mat. Palermo, II. Ser, 2022, 71, 525–535.
2. Atangana A., Baleanu D. New fractional derivatives with nonlocal and non-singular kernel: theory and application to heat transfer model. Therm. Sci., 2016, 20, 763–769.
3. Atangana A. On the new fractional derivative and application to nonlinear Fisher's reaction diffusion equation. Appl. Math. Comput., 2016, 273, 948–956.
4. Jafari H. A new general integral transform for solving integral equations. J. Adv. Res., 2021, 32, 133–138.