Boundedness of the L-index in a direction of the sum of slice holomorphic functions in the unit ball

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This report is based on the recent paper [2]. Our notations and definitions are taken from [2]. Let $\mathbf{b} = (b_1, \dots, b_n) \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ be a given direction, $\mathbb{B}^n = \{z \in \mathbb{C}^n \colon |z| < 1\}$ be a unit ball, $L\colon \mathbb{B}^n \to \mathbb{R}_+$ be a continuous function. The slice functions on S_z for fixed $z^0 \in \mathbb{B}^n$ we will denote as $g_{z^0}(t) = F(z^0 + t\mathbf{b})$ and $l_{z^0}(t) = L(z^0 + t\mathbf{b})$ for $t \in S_z$. Besides, we denote by $\langle a, c\rangle = \sum\limits_{j=1}^{n} a_j\overline{c_j}$ the Hermitian inner product in $\mathbb{C}^n$, where a, $c \in \mathbb{C}^n$.

Let $\widetilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{B}^n)$ be a class of functions which are holomorphic on every slices $\{z^0 + t\mathbf{b}\colon t \in S_{z^0}\}$ for each $z^0 \in \mathbb{B}^n$ and let $\mathcal{H}_{\mathbf{b}}(\mathbb{B}^n)$ be a class of functions from $\widetilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{B}^n)$ which are joint continuous. The notation $\partial_{\mathbf{b}}F(z)$ stands for the derivative of the function $g_z(t)$ at the point 0, i.e. for every $p \in \mathbb{N}$, $\partial_{\mathbf{b}}^p F(z) = g_z^{(p)}(0)$, where $g_z(t) = F(z + t\mathbf{b})$ is an analytic function of complex variable $t \in S_z$ for given $z \in \mathbb{B}^n$.

Together the hypothesis on joint continuity and the hypothesis on holomorphy in one direction do not imply holomorphy in whole n-dimensional unit ball. There were presented some examples to demonstrate it [1].

A function $F \in \widetilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{B}^n)$ is said [2] to be of *bounded L-index in the direction* $\mathbf{b}$, if there exists $m_0 \in \mathbb{Z}_+$ such that for all $m \in \mathbb{Z}_+$ and each $z \in \mathbb{C}^n$ inequality

$$\frac{|\partial_{\mathbf{b}}^m F(z)|}{m! L^m(z)} \le \max_{0\le k\le m_0} \frac{|\partial_{\mathbf{b}}^k F(z)|}{k! L^k(z)}$$

is true. For $z \in \mathbb{B}^n$ we denote

$$\lambda_{\mathbf{b}}(\eta) = \sup_{z\in\mathbb{B}^n} \sup_{t_1,t_2\in S_z} \left\{ \frac{L(z + t_1\mathbf{b})}{L(z + t_2\mathbf{b})} \colon |t_1 - t_2| \le \frac{\eta}{\min\{L(z + t_1\mathbf{b}), L(z + t_2\mathbf{b})\}} \right\}.$$

The notation $Q_{\mathbf{b}}(\mathbb{B}^n)$ stands for a class of positive continuous functions $L\colon \mathbb{B}^n \to \mathbb{R}_+$, satisfying for every $\eta \in [0, \beta]$ $\lambda_{\mathbf{b}}(\eta) < +\infty$ and for all $z \in \mathbb{B}^n$ one has $L(z) > \frac{\beta|\mathbf{b}|}{1-|z|}$, where $\beta > 1$ is some constant. The class of analytic functions of bounded index is not closed under the addition. The corresponding example was constructed by W. Pugh (see [4, 5]) in the case of entire function of single variable. A generalization of Pugh's example for entire functions of bounded L-index in direction is proposed in [3].

Let us consider an intersection of the hyperplane $\langle z, \mathbf{b}\rangle = 0$ with the unit ball. The intersection we denote by $A = \{z \in \mathbb{B}^n \colon \langle z, \mathbf{b}\rangle = 0\}$, where $\langle z, \mathbf{b}\rangle := \sum_{j=1}^n z_j b_j$. Obviously that $\bigcup_{z^0\in A}\{z^0 + t\mathbf{b}\colon |t| \le \frac{1-|z_0|}{|\mathbf{b}|}\} = \mathbb{B}^n$.

Let $z^0 \in A$ be a given point. If $F(z^0 + t\mathbf{b}) \not\equiv 0$ as a function of variable $t \in \mathbb{C}$, then there exists $t_0 \in S_{z^0}$ such that $F(z^0 + t_0\mathbf{b}) \ne 0$. We denote

$$B(z^0, t) = \left\{ t_0 \in S_{z^0} \colon |t_0 - t| < \min\left\{ \frac{\beta}{2L(z^0 + t\mathbf{b})}, \frac{1 - |z^0 + \mathbf{b}t|}{2|\mathbf{b}|} \right\}, F(z^0 + t_0\mathbf{b}) \ne 0 \right\},$$

$$B(z^0) = \bigcup_{|t|\le(1-|z^0|)/|\mathbf{b}|} B(z^0, t).$$

The following theorem is valid.

Theorem 1. *Let $L \in Q_{\mathbf{b}}(\mathbb{B}^n)$, $\alpha \in (1/\beta, 1)$ and F, $G \in \widetilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{B}^n)$, which satisfy condition:*
1) $G(z)$ has bounded L-index in the direction $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$;
2) for every $z = z^0 + t\mathbf{b} \in \mathbb{B}^n$, where $z^0 \in A$, and some $t_0 \in B(z^0, t)$, and $r = |t - t_0|L(z^0 + t\mathbf{b})$

$$\max\left\{|F(z^0 + t'\mathbf{b})| \colon |t' - t_0| = \frac{2r}{L(z^0 + t\mathbf{b})}\right\} \le \max_{0\le k\le N_{\mathbf{b}}(G_\alpha, L_\alpha, \mathbb{B}^n)} \left\{\frac{|\partial^k_{\mathbf{b}} G(z^0 + t\mathbf{b})|}{k!L^k(z^0 + t\mathbf{b})}\right\};$$

3) $c := \sup\limits_{z^0\in A} \dfrac{\max\left\{|F(z^0 + t'\mathbf{b})| \colon |t' - t_0| = \frac{2\lambda_2^{\mathbf{b}}(1)}{L(z^0+t_0\mathbf{b})}\right\}}{|F(z^0 + t_0\mathbf{b})|} < \infty$ where t_0 is chosen in 2).

If $|\varepsilon| \le \frac{1-\alpha}{2c}$, then the function $H(z) = G(z) + \varepsilon F(z)$ has bounded L-index in the direction $\mathbf{b}$ with $N_{\mathbf{b}}(H, L, \mathbb{B}^n) \le N_{\mathbf{b}}(G_\alpha, L_\alpha, \mathbb{B}^n)$, where $G_\alpha(z) = G(z/\alpha)$, $L_\alpha(z) = L(z/\alpha)$.

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