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The space of vector-valued strongly Lorentz sequences

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In this talk, we will introduce and examine the concept of strongly Lorentz sequence spaces. This concept is a broader extension of the sequence spaces first proposed by Cohen back in 1973. Additionally, we will explore the connections that exist between various classes of sequence spaces.

The letters E and F will always denote Banach spaces over $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$. The closed unit ball of E is denoted by B_E and its topological dual by E^*. By $\mathcal{L}(E,F)$ we denote the Banach space of all continuous linear operators $u: E \to F$ endowed with the usual sup norm. The symbol $E \equiv F$ means that E and F are isometrically isomorphic and $E \cong F$ means that E and F are isomorphic. As usual $\ell_p(E)$ denotes the vector space of all absolutely p-summable sequences, for $1 \leq p < +\infty$ with the usual norm $\|\cdot\|_p$ and $\ell_\infty(E)$ is the space of the bounded sequences in E. The closed subspace of $\ell_\infty(E)$ composed by the sequences $(x_n)_n$ in E such that $\lim\limits_{n\to+\infty} x_n = 0$ is denoted by $c_0(E)$ and $c_{00}(E)$ represents the subspace of $c_0(E)$ formed by the sequences $(x_n)_n$ in E, for which there is a N_0 such that $x_n = 0$, for any $n \geq N_0$. The unit coordinate vector e_n in these sequence spaces is the sequence $e_n = (\delta_{n,j})_j$, where $\delta_{n,j} = 0$ if $j \neq n$ and $\delta_{n,j} = 1$ if $j = n$. We denote by $\ell_p^w(E)$ the space of all weakly p-summable sequences with the norm $\|(x_n)_n\|_p^w = \sup_{x^* \in B_{X^*}} \|(\langle x_n, x^*\rangle)_n\|_p$ and $\ell_p\langle E\rangle$ the space of all strongly p-summable sequences (Cohen strongly p-summable sequences, see [2]) such that $(\varphi_n(x_n))_n \in \ell_1$, for any $(\varphi_n)_n \in \ell_{p^*}^w(E^*)$, where p^* denotes the conjugate of p, i.e. $\frac{1}{p} + \frac{1}{p^*} = 1$.
Let $x = (x_n)_n$ be a sequence in $c_0(E)$. We say that the sequence $(\|x_n\|)_{n=1}^{+\infty}$ admits a non-increasing rearrangement if there is an injection $\Phi : \mathbb{N} \to \mathbb{N}$ such that $\|x_{\Phi(1)}\| \geq \|x_{\Phi(2)}\| \geq \ldots \geq 0$ and $\Phi^{-1}(n)$ is not empty whenever $x_n \neq 0$. Following [3], the Lorentz sequence spaces $\ell_{p,q}(E)$ consists of all sequences $x = (x_n)_n \in c_0(E)$ such that

$$\|(x_n)_n\|_{p,q} = \begin{cases} \left(\sum\limits_{n=1}^{+\infty}\left(n^{\frac{1}{p}-\frac{1}{q}}\left\|x_{\Phi(n)}\right\|\right)^q\right)^{\frac{1}{q}} & \text{for } \ 1 \leq p \leq +\infty,\ 1 \leq q < +\infty \\ \sup_n n^{\frac{1}{p}}\left\|x_{\Phi(n)}\right\| & \text{for } \ \ 1 \leq p < +\infty,\ q = +\infty \end{cases}$$

is finite, where $\left(\left\|x_{\Phi(n)}\right\|\right)_{n=1}^{+\infty}$ is a non-increasing rearrangement of $(\|x_n\|)_{n=1}^{+\infty}$. In particular, if $E = \mathbb{K}$, $\ell_{p,q}(\mathbb{K})$ is denoted by $\ell_{p,q}$. We know that $a_{E,n}(x) = \left\|x_{\Phi(n)}\right\|$, for any n, where $a_{E,n}(x)$ is the n-th approximation number of x defined by

$$a_{E,n}(x) = \inf\left\{\|x-u\|_\infty ; u \in c_{00}(E) \text{ and card } u < n\right\}.$$

For $1 \leq q \leq p \leq +\infty$, $(\ell_{p,q}(E), \|\cdot\|_{p,q})$ is a Banach space, but for $1 \leq p \leq q \leq +\infty$, it is a quasi-Banach space. A sequences $(x_n)_n$ in E is said to be weakly Lorentz sequence (or it said to be a weak $\ell_{p,q}$ sequence) if $(\varphi(x_n))_n \in \ell_{p,q}$, for any $\varphi \in E^*$. We denote the vector of all weakly Lorentz sequences in E by $\ell_{p,q}^w(E)$. This is a quasi Banach space, with the quasi norm

$$\|(x_n)_n\|_{p,q}^w = \sup_{\varphi \in B_{E^*}} \|(\varphi(x_n))_n\|_{p,q}.$$

Note that $\ell^w_{p,p}(E) = \ell^w_p(E)$ and $\ell^w_{p,\infty}(E) = \ell_{p,\infty}(E)$, for $1 \le p < +\infty$. Naturally enough then, the hypothesis $1 \le p < +\infty$ and $q = +\infty$ will be omnipresent in our considerations. We extend the notion of strongly p-summable sequence spaces introduced by Cohen in [2] to Lorentz sequences as follows.

Definition 1. Let $1 \le p, q \le \infty$. We say that $(x_n)_n$ in E is a strongly Lorentz sequence if the series $\sum\limits_{n=1}^{\infty} |\varphi_n(x_n)|$ converges for any $(\varphi_n)_n \in \ell^w_{p^*,q^*}(E^*)$. The space of all such sequences shall be denoted by $\ell^{lz}_{p,q}\langle E\rangle$.

Proposition 1. *The expression*

$$\|(x_n)_n\|^{lz}_{\langle p,q\rangle} := \sup_{\|(\varphi_n)_n\|^w_{p^*,q^*} \le 1} \sum_{n=1}^{\infty} |\varphi_n(x_n)|$$

is a norm that makes $\ell^{lz}_{p,q}\langle E\rangle$ a Banach space.

Remark 1. For $p = q$, we have $\ell^{lz}_{p,p}\langle E\rangle = \ell_p\langle E\rangle$.

Proposition 2. *Let $u \in \mathcal{L}(E,F)$. The induced map $\widehat{u} : \ell^{lz}_{p,q}\langle E\rangle \to \ell^{lz}_{p,q}\langle F\rangle$ given by $\widehat{u}((x_n)_n) = (u(x_n))_n$ is a continuous linear operator and $\|\widehat{u}\| = \|u\|$.*

Theorem 1. *Given $1 \le p \le \infty$ and $1 \le q < \infty$. We have*

$$\ell^{lz}_{p,q}\langle E\rangle \subset \ell_{p,q}(E) \subset \ell^w_{p,q}(E).$$

Remark 2. For $q = 1$, we get $\ell^{lz}_{p,1}\langle E\rangle \cong \ell_{p,1}(E)$.

Proposition 3. *If E is finite dimensional, then*

$$\ell^{lz}_{\langle p,q\rangle}\langle E\rangle \cong \ell_{p,q}(E),$$

and for any $(x_n)_n \in \ell_{p,q}(E)$

$$m^{-\frac{1}{q^*}} \|(x_n)_n\|^{lz}_{\langle p,q\rangle} \le \|(x_n)_n\|_{p,q} \le M \|(x_n)_n\|^{lz}_{\langle p,q\rangle}.$$

We show the duality between strongly Lorentz sequence space $\ell^{lz}_{p,q}\langle E\rangle$ and weakly Lorentz sequence space $\ell^w_{p,q}(E)$.

Theorem 2. *[1] (The topological dual of $\ell^w_{p,q}(E)$) Given $1 \le p \le \infty$ and $1 \le q < \infty$. Then, $\ell^w_{p,q}(E)^*$ is isometrically isomorphic to $\ell^{lz}_{p^*,q^*}\langle E^*\rangle$, where a sequence $(x^*_n)_n$ in $\ell^{lz}_{p^*,q^*}\langle E^*\rangle$ is identified with the linear functional f given by*

$$f((x_n)_n) = \sum_{n=1}^{\infty} \langle x_n, x^*_n\rangle \text{ for each } (x_n)_n \in \ell^w_{p,q}(E).$$

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