On improved discrete Hardy's inequality

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For $p > 1$, and a sequence $\{a_n\}_{n=1}^{\infty}$ of complex numbers the classical discrete Hardy's inequality ([2], Theorem 326) in one dimension states that

$$\sum_{n=1}^{\infty}\Big|\frac{1}{n}\sum_{k=1}^{n}a_k\Big|^p < \Big(\frac{p}{p-1}\Big)^p\sum_{n=1}^{\infty}|a_n|^p, \tag{1}$$

holds unless a_n is null. Here $p > 1$ is a real number, and the constant term $\big(\frac{p}{p-1}\big)^p$ associated with the inequality (1) is sharp.

In case of $p = 2$, the above inequality (1) becomes

$$\sum_{n=1}^{\infty}\Big|\frac{1}{n}\sum_{k=1}^{n}a_k\Big|^2 < 4\sum_{n=1}^{\infty}|a_n|^2, \tag{2}$$

holds unless a_n is null, and the constant term '4' associated with inequality (2) is best possible.

The inequality (2) further equivalent to the following

$$\sum_{n=1}^{\infty}|A_n - A_{n-1}|^2 \geq \frac{1}{4}\sum_{n=1}^{\infty}\frac{|A_n|^2}{n^2}, \tag{3}$$

where $A_n = \sum_{k=1}^{n} a_k$ such that $A = (A_n) \in C_c(\mathbb{N}_0)$, the space of finitely supported functions on $\mathbb{N}_0$ with the assumption that $A_0 = 0$.

In 2018, Keller, Pinchover and Pogorzelski [3] obtained a surprising result on the improvement of inequality (3), and proved the following inequality

$$\sum_{n=1}^{\infty}|A_n - A_{n-1}|^2 \geq \sum_{n=1}^{\infty}w_n|A_n|^2 > \frac{1}{4}\sum_{n=1}^{\infty}\frac{|A_n|^2}{n^2}, \tag{4}$$

where the weight sequence $\{w_n\}$ is optimal (in sense of *criticality*) and is defined as follows

$$w_n = 2 - \sqrt{1 - \tfrac{1}{n}} - \sqrt{1 + \tfrac{1}{n}} > \tfrac{1}{4n^2},\ n \in \mathbb{N}.$$

The point-wise improved discrete Hardy's inequality (4) in one dimension can be further extended to a generalized form. In fact we have the following result (see [1], Theorem 2.1).

Theorem 1. *Let $\{A_n\}$ be any sequence of complex numbers such that $A_n \in C_c(\mathbb{N}_0)$ with $A_0 = 0$ and $g = \{g_n\}_{n=1}^{\infty}$ be any strictly positive sequence of real numbers. Then the following inequality holds*

$$\sum_{n=1}^{\infty}w_n(\lambda, g)|A_n|^2 \leq \sum_{n=1}^{\infty}\frac{|A_n - A_{n-1}|^2}{\lambda_n}, \tag{5}$$

where $\lambda = \{\lambda_n\}_{n\geq 1}$ *such that* $\lambda_n > 0$, $n \in \mathbb{N}$, *and the sequence* $w_n(\lambda, g)$ *is defined as below*

$$w_n(\lambda, g) = \frac{1}{\lambda_n} + \frac{1}{\lambda_{n+1}} - \frac{g_{n-1}}{\lambda_n g_n} - \frac{g_{n+1}}{\lambda_{n+1} g_n}.$$

Further, if there exists a sequence of elements $\gamma^N \in C_c(\mathbb{N}_0)$ *such that* $\gamma^N \leq \gamma^{N+1}$ *with* $\gamma^N \to 1$ *as* $N \to \infty$ *pointwise, and*

$$\lim_{N\to\infty} \sum_{n=2}^{\infty} \frac{g_n g_{n-1}}{\lambda_n} |\gamma_n^N - \gamma_{n-1}^N|^2 = 0,$$

then $w_n(\lambda, g)$ *is optimal.*

The criteria for '*optimality*' (in fact *criticality*) of the generalized weight sequence $w_n(\lambda, g)$ for particular $\{g_n\}$ in the extended improved discrete Hardy's inequality has been obtained. Several consequences of this result are given. The presentation will be based on the author's latest contribution [1] as given below.

1. Das B., Manna A. On the improvements of Hardy and Copson inequalities. 'Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas', 2023, 117(2), 18 p.
2. Hardy G. H., Littlewood J. E. and Pólya G. Inequalities. — 2nd Edition, Cambridge University Press, 1967, 324 p.
3. Keller M., Pinchover Y. and Pogorzelski F. An improved discrete Hardy inequality. Amer. Math. Monthly, 2018, 125 (4), 347–350.