Strong flow modifications of stochastic flows

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Let $\psi = (\psi_{s,t} : -\infty < s \leq t < \infty)$ be a stochastic flow on a locally compact separable metric space (M, ρ). $\psi = (\psi_{s,t} : -\infty < s \leq t < \infty)$ is a family of measurable random mappings of M that satisfy almost surely the evolutionary property, for any sequence $t_1 < t_2 < \ldots < t_n$ mappings $\psi_{t_1,t_2}, \ldots, \psi_{t_{n-1},t_n}$ are independent, for any $s < t$ mappings $\psi_{s,t}$ and $\psi_{0,t-s}$ are identically distributed and for all $f \in C_0(M)$, $s \leq t$ and $x \in M$,

$$\lim_{(u,v)\to(s,t)} \sup_{y\in M} E(f(\psi_{u,v}(y)) - f(\psi_{s,t}(y)))^2 = 0,$$

$$\lim_{y\to x} E(f(\psi_{s,t}(y)) - f(\psi_{s,t}(x)))^2 = 0, \lim_{y\to\infty} E(f(\psi_{s,t}(y)))^2 = 0.$$

A fundamental result obtained in [1] states that the relation

$$P^{(n)}(x, B) = \mathbb{P}((\psi_{0,t}(x_1), \ldots, \psi_{0,t}(x_n)) \in B), n \geq 1, x \in M^n, B \in \mathcal{B}(M^n),$$

establishes a one-to-one correspondence between stochastic flows on M and consistence sequences $(P^{(n)} : n \geq 1)$ of coalescing Feller transition functions. The sequence $(P^{(n)} : n \geq 1)$ defines distributions of finite-point motions of the flow. We will be interested in modifications of stochastic flows that satisfy the strong evolutionary property. These are modifications ψ' of ψ such that for almost all $\omega \in \Omega$ and all $r \leq s \leq t$,

$$\psi'_{s,t}(\omega, \cdot) \circ \psi'_{r,s}(\omega, \cdot) = \psi'_{r,t}(\omega, \cdot).$$

Our main result is the following. Let $((s_n, x_n) : n \geq 1)$ be a dense set in $\mathbb{R} \times M$, and $\Phi_n(t) = \psi_{s_n,t}(x_n)$, $t \geq s_n$, $n \geq 1$. Assume that with probability 1 for any compact $L \subset \mathbb{R} \times M$ the set

$$\{\Phi_n|_{[s\infty)} : s_n \leq s, (s, \Phi_n(s)) \in L\}$$

is relatively compact in the topology of uniform convergence on bounded intervals. Consider sets $\mathcal{K}_x^{s,t} = \bigcap_{\varepsilon>0} \overline{\{\Phi_n|_{[s,t]} : s_n \leq s, \rho(\Phi_n(s), x) \leq \varepsilon\}}$ and let

$$E = \{(s, x) \in \mathbb{R} \times M : \forall t > s \ \mathcal{K}_x^{s,t} \text{ contains at least two distinct functions}\}.$$

Assume that F is a closed subset of $\mathbb{R} \times M$, such that with probability 1 the set E is a subset of F.

Theorem 1. *If $\psi' = (\psi'_{s,t} : -\infty < s \leq t < \infty)$ is a modification of a stochastic flow ψ, such that with probability 1*

- *if $x = \Phi_n(s)$, then $\psi'_{s,\cdot}(x) = \Phi_n|_{[s,\infty)}$;*
- *if $r < s < t$, $\psi'_{r,s}(x) \in F$, $\psi'_{r,t}(x) \in F$, then there exists $n \geq 1$, such that $s_n \leq t$ and $\psi'_{r,\cdot}(x)|_{[t,\infty)} = \Phi_n|_{[t,\infty)}$.*

Then ψ' is a strong flow.

We also give examples of stochastic flows on metric graphs that satisfy conditions of the Theorem 1. In particular, we construct analogues of Arratia flows on metric graphs.

1. Le Jan Y., Raimond O. Flows, coalescence and noise. The Annals of Probability, 2004, Vol. 32, No. 2, 1247-1315.