Asymptotic behavior of the parameter estimators in the Vasicek model based on discrete observations

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We study the following mathematical model for the evolution of the interest rate, which was first proposed by O. Vasicek in 1977 [3]:

$$dX_t = (\alpha - \beta X_t)dt + \gamma dW_t, \quad X_t\big|_{t=0} = x_0 \in \mathbb{R}, \tag{1}$$

where the parameters $\alpha \in \mathbb{R}$, $\beta > 0$, and $\gamma > 0$ are fixed but unknown, and $W = \{W_t, t \ge 0\}$ is a Wiener process.

In the case of continuous-time observations of a trajectory of the process $X = \{X_t, t \in [0,T]\}$ the estimation procedure is well known. For such data the coefficient γ can be evaluated almost surely with the help of realized quadratic variations; α and β are then estimated using the maximum likelihood method or the least squares method, see, e.g., [1]. Our goal is to construct estimators for all parameters of the equation (1) for discrete-time observations.

We assume that we observe low-frequency data $X_0, X_h, X_{2h}, \dots, X_{nh}$ with a fixed positive step h. Let us introduce the following statistics:

$$\xi_n := \frac{1}{n}\sum_{k=0}^{n-1} X_{kh}, \qquad \eta_n := \frac{1}{n}\sum_{k=0}^{n-1} X_{kh}^2, \qquad \zeta_n := \frac{1}{n}\sum_{k=0}^{n-1} X_{kh}X_{(k+1)h}.$$

The construction of the estimators is based on the next lemma [2].

Lemma 1. *The following convergences hold:*

$$\xi_n \to \frac{\alpha}{\beta}, \qquad \eta_n \to \frac{\alpha^2}{\beta^2} + \frac{\gamma^2}{2\beta}, \qquad \zeta_n \to \frac{\alpha^2}{\beta^2} + \frac{\gamma^2}{2\beta}e^{-\beta h}$$

a.s., as $n \to \infty$.

The following estimators for the parameters α, β and γ are obtained:

$$\hat\beta_n = \frac{1}{h}\log^+ \frac{\eta_n - \xi_n^2}{\zeta_n - \xi_n^2}, \qquad \hat\alpha_n = \xi_n \hat\beta_n, \qquad \hat\gamma_n^2 = 2\hat\beta_n\left(\eta_n - \xi_n^2\right), \tag{2}$$

where

$$\log^+ x = \begin{cases} \log x, & x > 1, \\ 0, & x \le 1. \end{cases}$$

The following two theorems are the main results on the asymptotic behaviour of the estimators (2).

Theorem 1. *The following convergences hold a. s. as $n \to \infty$:*

$$\hat\alpha_n \to \alpha, \qquad \hat\beta_n \to \beta, \qquad \hat\gamma_n \to \gamma,$$

i.e., $\hat\alpha_n$, $\hat\beta_n$, $\hat\gamma_n$ are strongly consistent estimators of the parameters α, β, γ respectively.

Theorem 2. *The following convergence hold:*

$$\sqrt{n}\begin{pmatrix}\hat{\alpha}_n-\alpha\\ \hat{\beta}_n-\beta\\ \hat{\gamma}_n^2-\gamma^2\end{pmatrix}\to\mathcal{N}(0,\Sigma)$$

in distribution as $n\to\infty$, i.e., $\hat{\alpha}_n$, $\hat{\beta}_n$, $\hat{\gamma}_n$ are asymptotically normal estimators of the parameters α, β, γ respectively.

Moreover, the explicit formulas for the elements of asymptotic covariance matrix Σ are derived.

Finally, the quality of the estimators is illustrated with the help of simulated trajectories of the process X (generated as a solution of the stochastic differential equation (1) by the Euler–Maruyama method). The sample means and standard deviations of the estimates are reported in Table 1. The simulation study confirms the theoretical results stated in Theorems 1 and 2.

Table 1: Estimation results of the sampling interval $h=1$

	$n=100$	$n=500$	$n=1000$	$n=1500$	$n=2000$
Mean of $\hat{\alpha}_n$	1.0193	1.0683	1.0393	1.0280	1.0204
Mean of $\hat{\beta}_n$	1.9983	2.1324	2.0763	2.0555	2.0404
Mean of $\hat{\gamma}_n$	0.8896	1.0257	1.0168	1.0123	1.0095
S. Dev. of $\hat{\alpha}_n$	0.6647	0.2278	0.1368	0.1073	0.0897
S. Dev. of $\hat{\beta}_n$	1.2577	0.4404	0.2634	0.2059	0.1715
S. Dev. of $\hat{\gamma}_n$	0.4391	0.1073	0.0627	0.0505	0.0421

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