Operators of second quantization for Bernoulli noise

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In this work we construct an approximation of Gaussian white noise based on the sequence of Bernoulli random variables, develop the chaotic representation for functionals from Bernoulli noise, and construct an analog of the operators of second quantization for the Bernoulli case.

Let $\{\varepsilon_n, n \geq 1\}$ be a sequence of independent random variables with Bernoulli distribution:

$$\varepsilon_n = \begin{cases} 1, & \frac{1}{2} \\ -1, & \frac{1}{2} \end{cases}$$

Define $\varphi(f) := \sum_{k=1}^{n} f\left(\frac{k}{n}\right) \frac{\varepsilon_k}{\sqrt{n}}$, where $f \in C([0,1])$.

Definition 1. We will call a set of random variables $\{\varphi(f) : f \in C([0,1])\}$ by the Bernoulli noise in $C([0,1])$.

Theorem 1. *For all $f \in C([0,1])$, the following weak convergence holds:*

$$\varphi(f) \Longrightarrow N(0, \|f\|^2), n \to \infty$$

Remark 1. Similarly, for any $f_1, \ldots, f_n \in C([0,1])$:

$$(\varphi(f_1), \ldots, \varphi(f_k)) \Longrightarrow ((f_1, \xi), \ldots, (f_k, \xi)), \ n \to \infty,$$

where (f_i, ξ) is an element of Gaussian white noise.

Define the polynomials

$$A_k^n(\overrightarrow{\varepsilon}) = \frac{1}{n^{\frac{k}{2}}} \sum_{i_1 \neq \ldots \neq i_n} f\left(\frac{i_1}{n}, \ldots, \frac{i_k}{n}\right) \varepsilon_{i_1} \ldots \varepsilon_{i_k}, \ 1 \leq i_j \leq n.$$

Lemma 1. $\{A_k^n(\overrightarrow{\varepsilon}), 1 \leq k \leq n\}$ *is a system of orthogonal polynomials.*

Theorem 2. $\forall k \geq 1 :$

$$A_k^n(\overrightarrow{\varepsilon}) = \frac{1}{n^{\frac{k}{2}}} \sum_{i_1 \neq \ldots \neq i_n} f\left(\frac{i_1}{n}, \ldots, \frac{i_k}{n}\right) \varepsilon_{i_1} \ldots \varepsilon_{i_k} \Longrightarrow f(\underbrace{\xi, \ldots, \xi}_{k}), \ n \to \infty,$$

where $f(\underbrace{\xi, \ldots, \xi}_{k})$ is Hermite polynomial related to function f from definition for $A_k^n(\overrightarrow{\varepsilon})$.

Lemma 2. *Any function from $\{\varepsilon_n\}$ can be represented as*

$$\alpha(\overrightarrow{\varepsilon}) = \sum_{k=0}^{\infty} \sum_{i_1 \neq \ldots \neq i_k}^{n} a_{i_1 \ldots i_k} \varepsilon_{i_1} \ldots \varepsilon_{i_k}$$

Definition 2. Define the operator of second quantization corresponding to $\{p_n, n \geq 1\}, p_n \in (-1,1)$ as follows

$$\Gamma(p)\alpha = \alpha(p\varepsilon) = \sum_{k=0}^{\infty} \sum_{i_1 \neq \ldots \neq i_k}^{n} a_{i_1 \ldots i_k} p_{i_1} \varepsilon_{i_1} \ldots p_{i_k} \varepsilon_{i_k}.$$

Lemma 3.

$$\Gamma(p)\alpha = E(\alpha(\overrightarrow{\varepsilon}')|\overrightarrow{\varepsilon}),$$

where $\{\varepsilon_n'\}$ *is a sequence of independent random variables with the following distribution:*

ε_n' \ ε_n	*-1*	*+1*
-1	$\frac{1}{4} + \frac{1}{4}p_n$	$\frac{1}{4} - \frac{1}{4}p_n$
+1	$\frac{1}{4} - \frac{1}{4}p_n$	$\frac{1}{4} + \frac{1}{4}p_n$

Consider analogues of operators of the Ornstein-Uhlenbeck semigroup for Bernoulli noise. For any $n \geq 1$ consider $p_n = e^{-t}$. Then

$$T_t^{\varepsilon}\alpha = \Gamma(p)\alpha = \sum_{k=0}^{\infty} \sum_{i_1 \neq \ldots \neq i_k}^{n} a_{i_1 \ldots i_k} e^{-kt} \varepsilon_{i_1} \ldots \varepsilon_{i_k}.$$

Properties of the second quantization operator:

- $T_t^{\varepsilon} A_n^k \Longrightarrow e^{-kt} A_k(\xi, \ldots, \xi)$;
- $T_t^{\varepsilon} T_s^{\varepsilon} \alpha = \sum_{k=0}^{\infty} \sum_{i_1 \neq \ldots \neq i_k}^{n} a_{i_1 \ldots i_k} e^{-kt} e^{-ks} \varepsilon_{i_1} \ldots \varepsilon_{i_k} = T_{s+t}^{\varepsilon}\alpha$;
- $||T_t^{\varepsilon}\alpha||^2 \leq ||\alpha||^2$;
- $||T_t^{\varepsilon}\alpha - \alpha||^2 = \sum_{k=0}^{\infty} \sum_{i_1 \neq \ldots \neq i_k}^{n} a_{i_1 \ldots i_k}^2 (e^{-kt} - 1)^2 \longrightarrow 0, t \downarrow 0.$