Limit theorems for multifractal products of random fields

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Multifractal temporal and spatial data have been observed in many applications, for example, environmental processes (precipitations fields), engineering (teletraffic), and cosmology. Jaffard [3] showed that the class of multifractal random processes is wide and all Lévy processes except Brownian motion and Poisson processes are multifractal.

For multifractal signals a single exponent is not sufficient to characterise their scaling properties. The multifractal spectrum $D(h_0)$ of a signal shows Hausdorff dimensions of sets, where the Hurst exponent of a signal is h_0. Under so-called multifractal formalism [2], the Rényi function is linked to the multifractal spectrum by the Legendre transform, and is an important tool in the analysis of multifractal processes. The main focus of this investigation is to study the multifractal measure $\mu(\cdot)$ defined as a limit of measures $\mu_m(\cdot)$ generated by multifractal products of random fields.

$\boldsymbol{0}_n = (0,0,...,0)$ and $\boldsymbol{1}_n = (1,1,...,1)$ denote the origin and the unit vectors in $\mathbb{R}^n$ respectively. In what follows, $P_n[\boldsymbol{0}_n, \boldsymbol{t}]$ is the hyperparallelepiped with the opposite vertices $\boldsymbol{0}_n$ and $\boldsymbol{t} = (t_1, t_2, ..., t_n),\ t_i \in [0,1],\ i \in \overline{1,n}$, and the edges parallel to the axes. In the following, we assume that all random variables and random fields are defined on the same probability space $\{\Omega, \mathcal{F}, P\}$.

Let $\Lambda(\boldsymbol{s}),\ \boldsymbol{s} \in \mathbb{R}^n$, be a measurable, homogeneous and isotropic, nonnegative random field such that $P(\Lambda(\boldsymbol{0}_n) > 0) = 1$, $E\Lambda(\boldsymbol{0}_n) = 1$, and $E\Lambda^2(\boldsymbol{0}_n) < +\infty$. Let $\Lambda^{(i)}(\cdot),\ i \in 0,1,...$, be an infinite collection of independent stochastic copies of $\Lambda(\cdot)$. Let $b > 1$ be a scaling parameter. For $m \in \mathbb{N}$, the finite product $\Lambda_m(\cdot)$ of the random fields $\Lambda^{(i)}(\cdot)$ is defined by

$$\Lambda_m(\boldsymbol{s}) := \prod_{i=0}^{m-1} \Lambda^{(i)}(b^i \boldsymbol{s}).$$

The product $\Lambda_m(\cdot)$ can be used to define the nonnegative random measures $\mu_m(\cdot)$ on Borel subsets $B \subseteq P_n[\boldsymbol{0}_n, \boldsymbol{1}_n]$ as

$$\mu_m(B) := \int_B \Lambda_m(\boldsymbol{s}) d\boldsymbol{s},\ m \in \mathbb{N}.$$

Let $\mu(\cdot)$ be a random measure defined on Borel subsets of $P_n[\boldsymbol{0}_n, \boldsymbol{0}_n]$. The Rényi function of the random measure $\mu(\cdot)$ is a deterministic function given by

$$T(q) = \liminf_{j\to\infty} \frac{\log_2 E \sum_l \mu\left(B_l^{(j)}\right)^q}{\log_2 \left|B_0^{(j)}\right|} = \liminf_{j\to\infty} -\frac{\log_2 E \sum_l \mu(B_l^{(j)})^q}{nj},\ q > 0,$$

where $\{B_l^{(j)},\ l = 0,1,...2^{nj}-1,\ j = 1,2,...,\}$ denotes the mesh formed by the j-th level dyadic decomposition of $P_n[\boldsymbol{0}_n, \boldsymbol{1}_n]$.

Assumption 1 Let $\boldsymbol{p} = (p_1, p_2, ..., p_k),\ p_j \geq 0,\ j = \overline{1,k},\ k \geq 2$, and the function

$$\rho(\boldsymbol{u}_1, \boldsymbol{u}_2, ..., \boldsymbol{u}_k, \boldsymbol{p}) \geq E\left(\prod_{j=1}^{k} \Lambda^{p_j}(\boldsymbol{u}_j)\right),$$

for all $\boldsymbol{u}_j \in \mathbb{R}^n,\ j=\overline{1,k}$. Let also the function $\rho(\cdot)$ satisfy the condition

$$\rho(\boldsymbol{u}_1^{(1)}, \boldsymbol{u}_2^{(1)}, ..., \boldsymbol{u}_k^{(1)}, \boldsymbol{p}) \geq \rho(\boldsymbol{u}_1^{(2)}, \boldsymbol{u}_2^{(2)}, ..., \boldsymbol{u}_k^{(2)}, \boldsymbol{p})$$

if $||\boldsymbol{u}_i^{(2)} - \boldsymbol{u}_j^{(2)}|| \geq ||\boldsymbol{u}_i^{(1)} - \boldsymbol{u}_j^{(1)}||$, $\boldsymbol{u}_1^{(l)}, \boldsymbol{u}_2^{(l)}, ..., \boldsymbol{u}_k^{(l)} \in \mathbb{R}^n$, $i \neq j,\ i,l \in \{1,2,...,k\}$, $l=1,2$.

Theorem 1. *Let Assumption 1 hold true for the vector $\boldsymbol{p}=(p_1,p_2,...,p_k)$, $p_j \geq 1,\ j=\overline{1,k}$, such that $\sum_{j=1}^k p_j = p \geq 2$, and the scaling parameter b and the mixed moments satisfy the conditions*

$$b > (E\Lambda^p(\boldsymbol{0}_n))^{\frac{1}{n}}$$

$$\sum_{i=0}^{\infty} \ln\left(\rho(\boldsymbol{0}_n, b^i\boldsymbol{1}_n, 2b^i\boldsymbol{1}_n, ..., (k-1)b^i\boldsymbol{1}_n, \boldsymbol{p})\right) < \infty,$$

where $c\boldsymbol{1}_n = (c,c,...,c) \in \mathbb{R}^n$. Then, for a given finite or countable family of Borel sets $\mathfrak{B} = \{B_j:\ B_j \subseteq P_n[\boldsymbol{0}_n, \boldsymbol{1}_n]\}$, $\lim\limits_{m\to\infty} \mu_m(B_j) = \mu(B_j)$ in $L_q,\ q\in[0,p]$ and a.s.

Theorem 2. *Let Assumption 1 hold true and $\rho(\cdot, \boldsymbol{1}_p) \geq 1$, where p is an even integer. Also, let there exist such x_0 and $C_1 > 0$ that for all $x \geq x_0$*

$$\ln(\rho(\boldsymbol{1}_n, x\boldsymbol{1}_n, ..., (p-1)x\boldsymbol{1}_n, \boldsymbol{1}_p)) \leq C_1 x^{-\alpha},\ \alpha > n.$$

If

$$b > (E\Lambda^p(\boldsymbol{0}_n))^{1/n},$$

then for any Borel set B and $q\in[2,p]$

$$E|\mu(B) - \mu_m(B)|^q \leq C\Big(Leb(B)\Big)^{q(1-1/p)} \left(\frac{E\Lambda^p(\boldsymbol{0}_n)}{b^n}\right)^{\frac{mq}{p}}.$$

Theorem 3. *Let the conditions of Theorem 1 hold true for all $q \leq p$, i.e. $\mu(\cdot) \in L_q,\ q \geq p$. If for $q\in(0,1)$ the function $\rho(\boldsymbol{0}_n, \boldsymbol{x}, \boldsymbol{q})$, $\boldsymbol{q}=(q-1,1)$, is nondecreasing in $||\boldsymbol{x}||$, satisfies*

$$\sum_{i=1}^{\infty} \ln\left(\frac{\rho(\boldsymbol{0}_n, b^{-i}\boldsymbol{1}_n, \boldsymbol{q})}{E\Lambda^q(\boldsymbol{0}_n)}\right) < \infty,$$

and for $q \geq 1$ and some k-dimensional vector $\widetilde{\boldsymbol{p}} = (q/k, .., q/k)$ it holds

$$\sum_{i=1}^{\infty} \ln\left(\frac{E\Lambda^q(\boldsymbol{0}_n)}{\rho(\boldsymbol{0}_n, b^{-i}\boldsymbol{1}_n, ..., b^{-i}(k-1)\boldsymbol{1}_n, \widetilde{\boldsymbol{p}})}\right) < \infty.$$

Then the limiting measure $\mu(\cdot)$ possesses the following Rényi function

$$T(q) = q - 1 - \frac{1}{n}\log_b E\Lambda^q(\boldsymbol{0}_n),\ q\in[0,p].$$

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