A limit theorem for a nested infinite occupancy scheme in random environment

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The classical infinite occupancy scheme is a model in which one throws balls to an infinite array of boxes $1, 2, \ldots$ and the probability a ball hits the box k is p_k. Features of the occupancy pattern emerging after the first n balls are thrown have been intensively studied.

There is also a randomized version of the classical infinite occupancy scheme, in which the hitting probabilities of boxes are positive random variables $(P_k)_{k\in\mathbb{N}}$ with an arbitrary joint distribution satisfying $\sum_{k\in\mathbb{N}} P_k = 1$ almost surely (a.s.). We consider here a variant of this occupancy scheme, which corresponds to a nested family of boxes. The construction is conveniently described in terms of the genealogical structure of populations. Let $\mathcal{I}_0 := \{\varnothing\}$ be the initial ancestor and $\mathcal{I}_1 := \{1, 2, \ldots\}$ be the set of the first generation boxes with some random hitting probabilities $P_1, P_2, \ldots$. Divide now each box i into subboxes $i1, i2, \ldots$ and define the hitting probabilities of the subboxes by

$$P(ik) = P_i P_k^{(i)} \quad \text{for } k \in \mathbb{N},$$

where $(P_k^{(i)})_{k\in\mathbb{N}}$ is an independent copy of $(P_k)_{k\in\mathbb{N}}$. These subboxes are interpreted as the second generation boxes which form the set $\mathcal{I}_2$. We repeat this procedure for boxes of each generation until an ∞-ary tree of nested boxes $\cup_{k\in\mathbb{N}_0}\mathcal{I}_k$ has been constructed. Here, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

We assume that the random probabilities of boxes and the outcome of throwing balls are defined on a common probability space. For $n, j, r \in \mathbb{N}$, denote by $K_{n,j,r}$ the number of the jth generation boxes $v \in \mathcal{I}_j$ containing exactly r out of n first balls. Then the whole occupancy pattern in j-th generation is expressed by

$$K_{n,j}(u) := \sum_{r=\lceil n^{1-u}\rceil}^{n} K_{n,j,r}, \quad u \in [0, 1],$$

where $\lceil \cdot \rceil$ is the ceiling function.

For the given fragmentation law $(P_k)_{k\in\mathbb{N}}$, put $N(t) := \sum_{k\geq 1} \mathbb{1}_{\{-\log P_k \leq t\}}$, $V(t) = \mathbb{E}N(t)$ for $t \geq 0$ and $V_j(t) := \sum_{v\in\mathcal{I}_j} \mathbb{P}\{-\log P(v) \leq t\}$ for $j \in \mathbb{N},\ t \geq 0$.

Assume the following hold.

$$V(t) \ \sim\ t^{\alpha}\ell(t), \quad t \to \infty, \tag{1}$$

for some $\alpha \geq 0$ and some ℓ slowly varying at ∞;

$$\sup_{t\geq 1} \frac{\mathbb{E}(N(t))^2}{(V(t))^2} < \infty \tag{2}$$

and

$$\Big(\frac{N(ut)}{V(t)}\Big)_{u\geq 0} \ \Rightarrow\ (W(u))_{u\geq 0}, \quad t \to \infty, \tag{3}$$

where $\Rightarrow$ means weak convergence in the J_1-topology on Skorokhod space $D[0, \infty)$ and $(W(u))_{u\geq 0}$ is an a.s. locally Hölder continuous process with exponent $\beta > 0$.

Theorem 1. *Suppose (1), (2) and (3). Then*

$$\Big(\Big(\frac{K_{n,j}(u)}{(\log n)^{\alpha j}(\ell(\log n))^j}\Big)_{u\in[0,\,1]}\Big)_{j\in\mathbb{N}} \Rightarrow \Big(c_{j-1}(W_j(u))_{u\in[0,\,1]}\Big)_{j\in\mathbb{N}}, \quad n\to\infty.$$

in the product J_1-topology on $D[0,1]^{\mathbb{N}}$, where

$$c_j := \frac{(\Gamma(1+\alpha))^j}{\Gamma(1+\alpha j)}, \quad j\in\mathbb{N}_0,$$

Γ *is the Euler gamma function and*

$$W_j(u) := \int_{[0,\,u]} (u-y)^{\alpha(j-1)}\mathrm{d}W(y), \quad u\geq 0, j\in\mathbb{N}.$$