Limit theorems for dependent random variables with infinite means

Ismahen Bernou[1]

[1]Laboratoire de Statistique et Modélisations Aléatoires, Tlemcen, Algeria

ismahen825@gmail.com

We provide necessary and sufficient conditions for the convergence in probability of weighted averages of random variables with infinite means. Our results extend and improve the corresponding theorems obtained in the independent setup in [1] and [2].

Consider a sequence $\mathcal{X} = \{X_n, n \geq 1\}$ of real-valued random variables (rv's) defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and satisfying

$$\mathbb{P}(|X_j| > x) \asymp x^{-\alpha} \quad \text{for } j \geq 1 \text{ and some } \ 0 < \alpha \leq 1, \tag{1}$$

$$\limsup_{x\to\infty} \sup_{j\geq 1} x^{\alpha}\mathbb{P}(|X_j| > x) < \infty, \quad \text{for some } \ 0 < \alpha \leq 1 \tag{2}$$

and a Rosenthal-type maximal inequality, which means that there exist $p \geq 2$ and $C_p > 0$ such that for every $n \geq 1$, we have

$$\mathbb{E}\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} f_i(X_i)\right|^p\right) \leq C_p\left\{\left(\sum_{i=1}^{n}\mathbb{E}|f_i(X_i)|^2\right)^{p/2} + \sum_{i=1}^{n}\mathbb{E}|f_i(X_i)|^p\right\}, \tag{3}$$

whenever $f_1, f_2, \cdots, f_n$ are nondecreasing functions with $\mathbb{E}|f_i(X_i)|^p < \infty$ and $\mathbb{E}f_i(X_i) = 0$ for each $1 \leq i \leq n$.

The strong law of large numbers fails for these rv's since they have infinite means. Herein, we establish necessary and sufficient conditions for the convergence in probability of

$$W_n := \frac{1}{b_n}\max_{1\leq k\leq n}\left|\sum_{j=1}^{k} a_j(X_j - c_{nj})\right|,$$

for a suitable sequence $\{c_{nj},\ 1 \leq j \leq n\}$.

Theorem 1. *Let $0 < \alpha \leq 1$ and consider two sequences of positive constants $\{a_n,\ n \geq 1\}$ and $\{b_n,\ n \geq 1\}$ such that $\sum_{j=1}^{n} a_j^{\alpha} = o(b_n^{\alpha})$. If $\mathcal{X} = \{X_n, n \geq 1\}$ is a sequence of rv's satisfying (1), (2) and a Rosenthal-type maximal inequality (3), then $\{W_n, n \geq 1\}$ converges in probability.*

A necessary condition for the convergence in probability of $\{W_n, n \geq 1\}$ is also derived.

1. Adler A. An exact weak law of large numbers. Bull. Inst. Math. Acad. Sin, 2012, 7, 3, 417–422.
2. Nakata T. Weak laws of large numbers for weighted independent random variables with infinite mean. Stat. Probab. Lett., 2016, 109, 124–129.