Asymptotic properties of parameter estimators in mixed fractional stochastic heat equation

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We consider the following stochastic heat equation

$$\left(\frac{\partial u}{\partial t} - \frac{1}{2} \cdot \frac{\partial^2 u}{\partial x^2}\right)(t,x) = \sigma \dot{B}_x^H + \kappa \dot{W}_x, \quad t>0,\ x\in\mathbb{R}, \quad u(0,x)=0. \tag{1}$$

The right-hand side of (1) is a *mixed fractional noise*. It consists of two independent stochastic processes, namely, a fractional Brownian motion $B^H = \{B_x^H, x\in\mathbb{R}\}$ with Hurst parameter $H\in(0,1)$ and a Wiener process $W=\{W_x, x\in\mathbb{R}\}$, independent of B^H; σ and κ are some positive coefficients.

Let G be Green's function of the heat equation, that is

$$G(t,x) = \begin{cases} \frac{1}{\sqrt{2\pi t}} \exp\left\{-\frac{x^2}{2t}\right\}, & \text{if } t>0,\\ \delta_0(x), & \text{if } t=0. \end{cases}$$

We define a solution to SPDE (1) in a mild sense as follows

$$u(t,x) = \sigma \int_0^t \int_{\mathbb{R}} G(t-s,x-y)\, dB_y^H\, ds + \kappa \int_0^t \int_{\mathbb{R}} G(t-s,x-y)\, dW_y\, ds. \tag{2}$$

We prove the stationarity and ergodicity of the solution $u(t,x)$ as a function of the spatial variable x by analyzing the behavior of the covariance function.

It is supposed that for fixed $t_1,\dots,t_n$ and fixed step $\delta>0$, the random field u given by (2) is observed at the points $x_k=k\delta$, $k=1,\dots,N$. So the observations have the following form:

$$\{u(t_i,k\delta),\ i=1,\dots,n,\ k=1,\dots,N\}.$$

The estimator of H is defined as

$$\widehat{H}_N = f^{(-1)}\left(\frac{t_3^{-3/2}V_N(t_3) - t_2^{-3/2}V_N(t_2)}{t_2^{-3/2}V_N(t_2) - t_1^{-3/2}V_N(t_1)}\right),$$

where $f^{(-1)}$ denotes the inverse function of

$$f(H) \coloneqq \begin{cases} \frac{t_3^{H-1/2} - t_2^{H-1/2}}{t_2^{H-1/2} - t_1^{H-1/2}}, & \text{if } H\neq\frac{1}{2},\\ \frac{\log t_3 - \log t_2}{\log t_2 - \log t_1} & \text{if } H=\frac{1}{2}. \end{cases}$$

and

$$V_N(t) = \frac{1}{N}\sum_{k=1}^N u(t,k\delta)^2, \quad t>0,\ N\in\mathbb{N}.$$

Theorem 1. *1. For any $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$, $\widehat{H}_N$ is a strongly consistent estimator of the parameter H as $N \to \infty$.*

2. For $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, \frac{3}{4})$, the estimator $\widehat{H}_N$ is asymptotically normal:

$$\sqrt{N}\left(\widehat{H}_N - H\right) \xrightarrow{d} \mathcal{N}(0, \varsigma^2) \quad \text{as } N \to \infty,$$

where

$$\varsigma^2 = \frac{1}{D^2\sigma^4 c_H^2} \sum_{i,j=1}^{3} r_{t_i t_j}(H) L_i L_j,$$

$$L_1 = \frac{t_3^{H-\frac{1}{2}} - t_2^{H-\frac{1}{2}}}{t_1^{3/2}}, \quad L_2 = \frac{t_1^{H-\frac{1}{2}} - t_3^{H-\frac{1}{2}}}{t_2^{3/2}}, \quad L_3 = \frac{t_2^{H-\frac{1}{2}} - t_1^{H-\frac{1}{2}}}{t_3^{3/2}},$$

$$D = \left(t_2^{H-\frac{1}{2}} - t_1^{H-\frac{1}{2}}\right)\left(t_3^{H-\frac{1}{2}} \log t_3 - t_2^{H-\frac{1}{2}} \log t_2\right) - \left(t_3^{H-\frac{1}{2}} - t_2^{H-\frac{1}{2}}\right)\left(t_2^{H-\frac{1}{2}} \log t_2 - t_1^{H-\frac{1}{2}} \log t_1\right),$$

$$c_H = \frac{2^{H+1}(2^H - 1)\Gamma(H + \frac{1}{2})}{\sqrt{\pi}(H+1)}, \quad r_{t_i t_j}(H) = 2 \sum_{k=-\infty}^{\infty} \operatorname{cov}\big(u(t_i, k\delta), u(t_j, 0)\big)^2.$$

Now we assume that the Hurst index H is known and investigate the estimation of the coefficients σ and κ:

$$\widehat{\sigma}_N^2 = \frac{t_1^{-3/2} V_N(t_1) - t_2^{-3/2} V_N(t_2)}{c_H\left(t_1^{H-1/2} - t_2^{H-1/2}\right)}, \quad \widehat{\kappa}_N^2 = \frac{t_1^{-1-H} V_N(t_1) - t_2^{-1-H} V_N(t_2)}{c_{\frac{1}{2}}\left(t_1^{1/2-H} - t_2^{1/2-H}\right)}.$$

Theorem 2. *1. For any $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$, $(\widehat{\sigma}_N^2, \widehat{\kappa}_N^2)$ is a strongly consistent estimator of the parameter (σ^2, κ^2) as $N \to \infty$.*

2. For $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, \frac{3}{4})$, the estimator $(\widehat{\sigma}_N^2, \widehat{\kappa}_N^2)$ is asymptotically normal:

$$\sqrt{N}\begin{pmatrix} \widehat{\sigma}_N^2 - \sigma^2 \\ \widehat{\kappa}_N^2 - \kappa^2 \end{pmatrix} \xrightarrow{d} \mathcal{N}(0, \Sigma) \quad \text{as } N \to \infty,$$

where the asymptotic covariance matrix Σ consists of the following elements:

$$\Sigma_{11} = \frac{t_1^{-3}(r_{t_1t_1}(H) + r_{t_1t_2}(H)) + t_2^{-3}(r_{t_1t_2}(H) + r_{t_2t_2}(H))}{c_H^2\left(t_1^{2H-1} - 2(t_1t_2)^{H-\frac{1}{2}} + t_2^{2H-1}\right)},$$

$$\Sigma_{12} = \Sigma_{21} = \frac{t_1^{-\frac{5}{2}-H}(r_{t_1t_1}(H) + r_{t_1t_2}(H)) + t_2^{-\frac{5}{2}-H}(r_{t_1t_2}(H) + r_{t_2t_2}(H))}{c_H c_{\frac{1}{2}}\left(2 - t_1^{H-\frac{1}{2}} t_2^{\frac{1}{2}-H} - t_1^{\frac{1}{2}-H} t_2^{H-\frac{1}{2}}\right)},$$

$$\Sigma_{22} = \frac{t_1^{-2-H}(r_{t_1t_1}(H) + r_{t_1t_2}(H)) + t_2^{-2-H}(r_{t_1t_2}(H) + r_{t_2t_2}(H))}{c_{\frac{1}{2}}^2\left(t_1^{1-2H} - 2(t_1t_2)^{\frac{1}{2}-H} + t_2^{1-2H}\right)}.$$

The quality of estimators is illustrated by simulation experiments.

1. D. Avetisian, K. Ralchenko, Parameter estimation in mixed fractional stochastic heat equation. Modern Stochastics: Theory and Applications, 2023, V. 10, No. 2, 175 – 195.