Resonance case for weakly perturbed linear boundary value problems for integro-differential equations

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In various applied sciences, mathematical models of processes are appearing, which are described by systems of algebraic and integro-differential equations. A wide range of such mathematical models are described by systems of integro-differential equations with various types of disturbances or nonlinearities. It is known that some problems of optimal control, linear programming, economics, theory of elasticity, hydrodynamics, chemical and biological kinetics, etc. are modeled by such operator equations.

In [5], it is proposed to consider the application of the theory of generalized inverse operators for the study of linear boundary value problems for systems of integro-differential equations in the noncritical case, as well as a weakly perturbed linear boundary value problem for such equations in the resonant case.

To investigate the existence of solutions to such problems as in [2], one can use the apparatus of the theory of pseudo-inverse matrices and operators, which was developed in the works of A.M. Samoilenko, A.A. Boichuk [1, 6] and in works [3]–[5].

Consider the boundary value problem for a linear system of integro-differential equations with a small parameter ε:

$$\dot{x}(t) - \Phi(t)\int_a^b \Big[A(s)x(s) + B(s)\dot{x}(s)\Big]ds = f(t) + \varepsilon\int_a^b \Big[K(t,s)x(s) + K_1(t,s)\dot{x}(s)\Big]ds, \tag{1}$$

$$\ell x(\cdot,\varepsilon) = \alpha + \varepsilon\ell_1 x(\cdot,\varepsilon) \ \in \ \mathbb{R}^q. \tag{2}$$

and we will look for the structure of the set of solutions of this problem in the space $D_2[a,b]$ of n-dimensional absolutely continuous differentiable vector functions: $x = x(t,\varepsilon) : x(\cdot,\varepsilon) \in D_2[a,b]$, $\dot{x}(\cdot,\varepsilon) \in L_2[a,b]$, $x(t,\cdot) \in C(0,\varepsilon_0]$. Here $A(t), B(t), \Phi(t)$ — $(m\times n)$, $(m\times n)$, $(n\times m)$-dimensional matrices whose components belong to the space $L_2[a,b]$; the column vectors of the matrix $\Phi(t)$ are linearly independent on $[a,b]$, $f(t)$ is an n-dimensional vector function from $L_2[a,b]$, $K(t,s), K_1(t,s)$ are $(n\times n)$-dimensional matrices, the components of which are defined in the space of segment-invariant functions from $L_2[a,b]$; ℓ, ℓ_1 are linear bounded q-dimensional vector functionals, $\alpha = col(\alpha_1,\alpha_2,...,\alpha_q) \in R^q$. Let us find out the bifurcation conditions of the solution of the noetherian $(n \neq q)$ boundary value problem (1), (2). Assume that the generating (for $\varepsilon = 0$) boundary value problem for (1), (2) is unsolvable for any inhomogeneities $f(t) \in L_2[a,b]$ and $\alpha \in \mathbb{R}^q$. That is, the solvability criterion [1] for such a boundary value problem is not fulfilled and the following inequalities hold $P_{D^*_{d_1}}\tilde{b} \neq 0$, $P_{Q^*_{d_2}}(\alpha - \ell F(\cdot)) \neq 0$, $d_1 = m - rankD$, $d_2 = q - rankQ$.

And, moreover, the known sufficient condition [5] for the solvability of a weakly perturbed boundary value problem is also not fulfilled, then the solution $x = x(t,\varepsilon)$ of the boundary value problem (1), (2) in the form of a part the Laurent series $x(t,\varepsilon) = \sum_{k=-1}^{\infty}\varepsilon^k x_k(t,c_k)$ does not exist. The following theorem is true.

Theorem 1. *Assume that the weakly perturbed boundary value problem (1), (2) satisfies the above conditions so that the corresponding generating boundary value problem (for $\varepsilon = 0$) is unsolvable under any inhomogeneities$f(t) \in L_2[a,b]$, $\alpha \in \mathbb{R}^p$. If condition*

$$P_{B_0^*} \begin{bmatrix} P_{D_{d_1}^*} \\ P_{Q_{d_2}^*} \end{bmatrix} = 0, \tag{3}$$

is satisfied, then the boundary value problem (1), (2) will have at least one solution specified in the class of vector functions $x = x(t,\varepsilon) : x(\cdot,\varepsilon) \in D_2[a,b], \quad \dot{x}(\cdot,\varepsilon) \in L_2[a,b], \quad x(t,\cdot) \in C(0,\varepsilon_0]$ in the form of a series $x(t,\varepsilon) = \sum\limits_{k=-2}^{\infty} \varepsilon^k x_k(t,c_k)$, that converges for a fixed $\varepsilon \in (0;\varepsilon_*]$.

And constituent components of this series are determined by the iterative process in [2]. Here, the well-known [3] matrix B_0 is $((d_1+d_2)\times r_2)$dimensional and is constructed from the components of the boundary value problem (1), (2), matrices P_{B_0}, $P_{B_0^*}$ are $(r_2 \times r_2)$-, $((d_1+d_2)\times(d_1+d_2))$-dimensional orthoprojectors, which translate spaces $\mathbb{R}^{r_2}$, $\mathbb{R}^{d_1+d_2}$ into the kernel and cokernel of matrix B_0, respectively.

Remark. If $P_{N(B_0)} = 0$, then at each step of the iteration process, the obtained operator equations will be n-normal and uniquely solvable. Then, if the condition (3) is fulfilled, the boundary value problem (1), (2) will have a unique solution.

The resulting technique for studying linear boundary value problems for integro-differential equations, developed in this work [2], can be successfully applied to a wide class of problems, if their linear part is an operator that does not have an inverse.

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