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On the stability of finite difference schemes for nonlinear diffusion equations

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The Complex diffusion is a denoising procedure commonly used in image processing, such as noise removal, retouching, stereo vision or optical flow.

In particular, nonlinear complex scattering has proven to be a well-conditioned numerical technique that has been successfully applied in medical imaging.

The stability properties of a class of finite difference schemes for the complex nonlinear diffusion equation were studied by Araùjo.A and al. 2012. [1,2], where only explicit and implicit schemes were considered and no reaction terms were considered.

Soit $\Omega \subset \mathbb{R}^d$, $d \geqslant 1$, the Cartesian product of open intervals of $\mathbb{R}$, with the boundary $\Gamma = \partial\Omega$,

$$\Omega = \prod_{j=1}^{d}]a_j, b_j[\,,$$

with $a_j, b_j \in \mathbb{R}$. Let $Q = \Omega \times [0,T]$, with $T > 0$ and $u{:}\overline{Q} = \overline{\Omega} \times [0,T] \to C$. We consider a nonlinear diffusion process with a coefficient D non-constant complex $D(x,t,u) = D_R(x,t,u) + iD_I(x,t,u)$, where $D_R(x,t,u)$ and $D_I(x,t,u)$ are real functions. We must also assume that [1]

$$D_R(x,t,u) \geqslant 0, \qquad (x,t) \in \overline{Q}, \tag{1}$$

and there is a constant $L > 0$ such that

$$0 < |D(x,t,u)| \leqslant L, \quad (x,t) \in \overline{Q}. \tag{2}$$

We define the initial boundary value problem for the unknown function $u(x,t)$

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) = Div\left(D(x,t,u)\nabla u(x,t)\right), & (x,t) \in Q, \\ u(x,0) = u^0(x), & x \in \overline{\Omega}, \\ \alpha u(x,t) + \beta \frac{\partial u}{\partial \nu}(x,t) = 0, \quad x \in \Gamma, & t \in [0,T], \end{cases} \tag{3}$$

where $\frac{\partial u}{\partial \nu}$ denotes the derivative in the direction of the exterior normal Ω on Γ.

For the boundary conditions, we assume that

$$\alpha\beta = 0 \qquad et \qquad \alpha + \beta \neq 0.$$

We construite a mesh of $\overline{Q}$. For the time interval we consider the mesh [2]

$$0 = t^0 < t^1 < \cdots < t^{(M-1)} < t^M = T,$$

where $M \geq 1$ is an integer $t^{m+1} - t^m = \Delta t^m$, $m = 0, ..., M-1$.

Let h_k the mesh in the $k - i\grave{e}me$ spatial coordinate, such that $h_k = \frac{b_k - a_k}{N_k}$, for $k = 1, ..., d$, and $N_k \geq 2$ is an integer. Let $h = \max h_k$ et $k = \max \Delta t^m$. The set of points

$$x_j = (a_1 + j_1 h_1, ..., a_d + j_d), 0 \leq j_k \leq N_k, k = 1, ..., d,$$

defines a grid in space that we denote by $\overline{\Omega}_h$. We associate to the point (x_j, t^m) the coordinates $(j, m) = (j_1, ..., j_d, m)$.

We define a mesh of $\overline{Q}$, denoted by $\overline{Q_h^{\Delta t}}$, by the Cartesian product of the grid of space $\overline{\Omega}_h$ and a grid in the spatio-temporal domain. Let $Q_h^{\Delta t} = \overline{Q_h^{\Delta t}} \cap Q$ and $\Gamma_h^{\Delta t} = \overline{Q_h^{\Delta t}} \cap \Gamma \times [0, T]$.

We note V_j^m the value of a function V, defined in $\overline{Q_h^{\Delta t}}$, at point (x_j, t^m).

We define the progressive and regressive finite difference operators at the point (x_j, t^m) à the $k - i\grave{t}hth$ spatial coordinate,

$$\delta_k^+ V_j^m = \frac{V_{j+e_k}^m - V_j^m}{h_k}, \qquad \delta_k^- V_j^m = \frac{V_j^m - V_{j-e_k}^m}{h_k},$$

where e_k represents the $k - i\acute{t}hth$ element of the canonical basis of $\mathbb{R}^d$.

The finite difference scheme approximating (3) in $\widetilde{Q_h^{\Delta t}}$ is:

$$\begin{cases} \frac{U_j^{m+1} - U_j^m}{\Delta t} = \sum_{k=1}^d \delta_k^+ \left(D_{j-(1/2)e_k}^{m+\theta} \delta_k^- U_j^{m+\theta} \right), \text{ sur } \widetilde{Q_h^{\Delta t}}, \\ U_j^0 = u_0(x_j), \qquad \text{sur } \overline{\Omega}_h, \\ \alpha U_j^m + \frac{\beta}{2} \sum_{k=1}^d (\delta_k^+ U_j^m + \delta_k^- U_j^m).\nu_k = 0, \quad \text{sur } \Gamma_h^{\Delta t}, \end{cases} \tag{4}$$

where $V_j^{m+\theta} = \theta V_j^{m+1} + (1-\theta) V_j^m$, $\theta \in [0, 1]$, U_j^m represents the approximation of $u(x_j, t^m)$ and

$$D_{j-(1/2)e_k}^m = \frac{D(x_j, t^m, U_j^m) + D(x_{j-e_k}, t^m, U_{j-e_k}^m)}{2}.$$

The stability of the finite difference scheme (3). In the following theorem, we give the stability conditions for the $\theta - schema$ [2].

Theorem 1. *Assume that the conditions (1) and (2) hold. If $\theta \in [\frac{1}{2}, 1]$ then the method (4) is unconditionally stable. If $\theta \in [0, \frac{1}{2}[$ then the schema (4) is stable if the condition*

$$\Delta t \leq \frac{(\min\{h_1, ..., h_d\})^2}{2d(1-2\theta) \max\limits_{x_j \in \overline{\Omega}_h} \frac{\left| D_j^{m+\theta} \right|^2}{D_{R_j}^{m+\theta}}}, \qquad m = 1, ..., M-1,$$

holds, provided there is some ξ such that

$$0 < \xi \leq D_{R_j}^{m+\theta} \qquad \forall j, m \in Q_h^{\Delta t}.$$

1. Araùjo A., Barbeiro S., Serranho P. Stability of finit difference schemes for complex diffusion processes, SIAM J. Num. Anal., 2012, 50, 1284-1296.
2. Araùjo A., Barbeiro S., Serranho P. Finite difference schemes for nonlinear complex reaction-diffusion processes. SIAM J. Num-Anal., 2014.