Derivation and analysis of a Fokker-Planck equation describing a population of spiking resonate-and-fire neurons

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Based on Izhikhevich's resonate-and-fire model [1] of the temporal evolution of membrane potential, we derive a kinetic Fokker-Planck equation describing the probability density $\rho(x, v, t)$ of finding neurons in a population having a potential x, with a time derivative v, at a given time t, other models having already been studied in, for example [2], [3], or [4].

The equation reads

$$\begin{array}{l}
\forall x \leq u_F, \frac{\partial p(x,v,t)}{\partial t} + \nabla(\mu p(x,v,t) - \overline{\overline{D}}\nabla p(x,v,t)) = \delta_{u_R}(x) \otimes \delta_0(v) N(t), \\
N(t) = \int_{v\in\mathbb{R}^+} v.p(u_F, v, t)dv, \\
\forall v, p(-\infty, v, t) = 0; \forall v < 0, p(u_F, v, t) = 0 \\
\forall x > u_F, \forall v, p(x, v, t) = 0, p^0(x, v) \geq 0, \int_{-\infty}^{u_F}\int_{-\infty}^{\infty} p^0(x, v)dxdv = 1
\end{array}$$

with $\mu(x, v, t) = \begin{pmatrix} v \\ -\omega_0^2 x - v/\tau + bN(t)/\tau + b\nu_{ext}/\tau \end{pmatrix}$ and $\overline{\overline{D}} = (a_0 + a_1 N(t)) \begin{bmatrix} 0 & 0 \\ 0 & 1/\tau^2 \end{bmatrix}$ $a_0, a_1, b, \nu, \omega_0, \tau$ being neurological constants.

Let us define the following open subset of $\mathbb{R}^2$: $\Omega = (-\infty, u_F) \times \mathbb{R}$. Now let us first define some notions of solutions that will allow us to work on the problem:

Definition 1. A pair of functions (p, N) is said to be a strong solution of the system (P) on $[0; T[, T \in \mathbb{R}_+^* \cap \infty$ when :

- $p \in \mathcal{C}^0(]-\infty, u_F] \times]-\infty, \infty[\times) \cap \mathcal{C}^{2,2,1}(]-\infty, u_R[\cup]u_R, u_F] \times]-\infty, \infty[\times[0, T[) \cap L^\infty([0, T[, L^1(]-\infty, u_F] \times]-\infty, \infty[))$ and $N \in \mathcal{C}^0([0, T[)$
- The functions p and N are solutions of (P) in the the classical sense on $]-\infty, u_R[\cup]u_R, u_F] \times]-\infty, \infty[$ and in the sense of distributions on $]-\infty; u_F] \times]-\infty, \infty[$.

Definition 2. A pair of non-negative functions (p, N) with $p \in L^\infty(\mathbb{R}^+; L^2_+((-\infty, u_F) \times]-\infty, \infty[)), N \in L^1_{loc,+}(\mathbb{R}^+)$ is a weak solution of the problem (P) if for any test function $\phi((x, v), t) \in \mathcal{C}^\infty(]-\infty; u_F] \times]-\infty, \infty[\times[0; T])$ such that $v\frac{\partial\phi}{\partial x}, (x+v)\frac{\partial\phi}{\partial v} \in L^\infty(]-\infty; u_F[\times]-\infty, \infty[\times]0; T[)$, we have

$$\begin{array}{l}
\int_0^T \int\limits_{x=-\infty}^{u_F} \int\limits_{v=-\infty}^{\infty} p(x, v, t)[-\frac{\partial\phi}{\partial t} - \mu(x, v, t)\nabla\phi - \nabla(\overline{\overline{D}}(t)\nabla\phi)]\, \mathrm{d}x\mathrm{d}v\mathrm{d}t = \\
\int_0^T N(t)(\phi(u_R, 0, t) - \phi(u_F, 0, t))\, \mathrm{d}t + \int\limits_{-\infty}^{u_F} \int\limits_{-\infty}^{+\infty} p^0(x, v)\phi(x, v, 0)\, \mathrm{d}x\mathrm{d}v \\
- \int\limits_{-\infty}^{u_F} \int\limits_{-\infty}^{+\infty} p(x, v, T)\phi(x, v, T)\, \mathrm{d}x\mathrm{d}v
\end{array}$$

We could obtain the following theorem:

Theorem 1. *Let p^0 be in $L^2(\mathbb{R}^2 \times \mathbb{R}^+; \mathbb{R})$, and (p, N) a weak solution of our problem (P), with p^0 verifying the following conditions, denoted (C) : $\int\limits_{(x,v)\in\mathbb{R}^2} p^0(x,v)\,\mathrm{d}x\mathrm{d}v = 1$; $\forall (x,v) \in \mathbb{R}^2$, $p^0(x,v) \geq 0$; $p^0 = 0$ a.e. on $\partial\Omega_{hyp}$ and p being possibly negative for $t > 0$ meanwhile $\forall t \geq 0, N(t) \geq 0$. Then : $\forall t \geq 0$, $\int\limits_{(x,v)\in\Omega} p(x,v,t)\,\mathrm{d}x\mathrm{d}v = 1$ and $\forall t \geq 0$, $p(x,v,t) \geq 0$ a.e. in Ω*

Even though we could obtain some numerical approximations of the solutions, the classical toolbox associated with linear operators cannot be used because of the non-local linearity due to the function N, which makes the theoretical study complicated. Thus, to obtain more information, we then focus on a linearized version of the equation around 0, tackling the study of a boundary value Cauchy problem for an hypoelliptic operator, using different methods and approaches. In this linearized version, the function N disappears from the drift vector $\boldsymbol{\mu_L}$ and the diffusion matrix.

On top of the degeneracy of the problem, the difficulties associated with the right-hand side of the equation, which displays a non-local dependency on the solution as well as a measure, are of course of prime concern.

We will present steps towards existence and uniqueness in this difficult case by simplifying this right-hand side and imposing some boundedness on the spatial variable x, using the framework developped in [5] :

Theorem 2. *If we consider $\Omega =]x_{min}; x_{max}[\times\mathbb{R}$, noting Q_T the classical parabolic cylinder associated with Ω, if p_0 is in $L^2(\Omega)$, the right-hand side f is in $L^2(Q_T)$ we have existence and uniqueness of a solution to the linearized problem for all initial conditions verifying (C), the solution being in the space $Y = \{p \in H : \mathcal{T}p \in H'\}$, with $H = \{p \in L^2(Q_T) : \nabla_v p \in L^2(Q_T)\}$ and $\mathcal{T}p = \partial_t p + \mu_L.\nabla p$.*

As well as another result, this time in the unbounded setting, displaying similarities but still with radically different methods and point of view in the line of the work in [6] and [7].

Theorem 3. *For any boundary condition in $C_0(\partial\Omega) \times \mathbb{R}$, there exists an unique weak solution p verifying $p \in L^2_{v,t}(\Omega_v \times]0, T[, H^1_x(\Omega_x))$ and $\partial_t p + \mu_L.\nabla p \in L^2_{v,t}(\Omega_v \times]0, T[, H^{-1}_x(\Omega_x))$.*

Some associated results (conservation of positivity, convergence of the transport trajectories in the Wasserstein metric) which allow for a better understanding of the problem, will also be presented.

1. Izhikevich E.M. Resonate-and-fire neurons. Neural Networks, 14(6), 2001, 883–894.
2. Brunel N. Dynamics of sparsely connected networks of excitatory and inhibitory spiking neurons, Journal of Computational Neuroscience, 2000.
3. Cáceres M.J., Carrillo J.A., Perthame B. Analysis of nonlinear noisy integrate and fire neuron models: blow-up and steady states. The Journal of Mathematical Neuroscience, 1 (1):7 2011.
4. Torres, N., Perthame B., Salort D. A multiple time renewal equation for neural assemblies with elapsed time model, Nonlinearity, 35, 2021, 5051.
5. Carrillo, J.A. Global weak solutions for the initial boundary value problems Vlasov Poisson Fokker Planck System. Mathematical Methods in the Applied Sciences, 1998.
6. Litsgard M., Nystrom K. Potential theory for a class of strongly degenerate parabolic operators of Kolmogorov type with rough coefficients. Journal de Mathématiques Pures et Appliquées, 2022.
7. Manfredini M. The Dirichlet problem for a class of ultraparabolic equations. Advances in Differential Equations, 1997.