Investigation of Commutative Properties of Discontinuous Galerkin and backward Euler Methods in PDE Constrained Optimal Control Problems

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Let us consider the following bi-objective constrained optimal control problem

$$\min_{(u,y)} \left\{ \frac{1}{2}\int_0^T \int_\Omega [y(t,x)-y_d(t,x)]^2\, dx\, dt,\ \frac{1}{2}\int_0^T ||u(t)||^2 dt \right\}, \tag{1}$$

$$\begin{cases} \dfrac{\partial y}{\partial t}(t,x) - k\,\Delta y(t,x) + \beta(t,x)\cdot\nabla y(t,x) = \displaystyle\sum_{i=1}^m u_i(t)\,\chi_i(x) & \text{for} \quad (t,x)\in[0,T]\times\Omega, \\ \dfrac{\partial y}{\partial \eta}(t,x) + \alpha_j\, y(t,x) = \alpha_j\, y_a(t) & \text{for} \quad (t,x)\in\Sigma_j=(0,T)\times\Gamma_j, j\in\{1,\cdots,s\}, \\ y(0,x)=y_0(x) & \text{for} \quad x\in\Omega, \end{cases} \tag{2}$$

$$u_a(t) \le u(t) \le u_b(t), \quad \text{for almost} \quad t\in[0,T], \tag{3}$$

where,

- The diffusion parameter k is a positive constant.
- The convection term $\beta(t,x)$ is time-dependent and satisfy $\beta \in B := L^\infty\big(0,T; L^\infty(\Omega;\mathbb{R}^n)\big)$.
- The initial temperature is supposed to fulfil $y_0 \in L^2(\Omega)$ and the outer temperature $y_a \in L^2(0,T)$ is space-independent.
- The bounds in the bilateral box constraints fulfil $u_a, u_b \in L^\infty(0,T;\mathbb{R}^m)$.

The bi-objective optimization problem $(1)-(3)$ can be transformed, using the weighted sum method, to a single-objective optimization problem where the objective function is

$$J(u,y) = \frac{w}{2}||y-y_d||^2_{L^2(0,T;L^2(\Omega))} + \frac{1-w}{2}||u||^2_{L^2(0,T;\mathbb{R}^m)}, \quad w\in]0,1[. \tag{4}$$

For space discretization, we use the symmetric interior penalty Galerkin (SIPG) discretization for the diffusion and an upwind discretization for the advection part.

$$(\partial_t y_h, v_h) + a_h^s(y_h,v_h) = (f_h, v_h), \quad \forall v_h \in V_h,$$

where

$$\begin{aligned} a_h^s(y,v) \;=\; & \sum_{T\in T_h}\int_T k\nabla y\cdot\nabla v\,dx - \sum_{E\in\varepsilon_h}\int_E \{\{k\nabla y\}\}\cdot[[v]]\,ds - \sum_{E\in\varepsilon_h}\int_E \{\{k\nabla v\}\}\cdot[[y]]\,ds \\ & + \sum_{E\in\varepsilon_h}\frac{\delta_\varepsilon}{h_E}\int_E [[y]]\cdot[[v]]\,ds + \sum_{T\in T_h}\int_T (\beta\cdot\nabla y v)\,dx + \sum_{T\in T_h}\int_{\partial T^-|\Gamma_j} \beta\cdot n(y^e-y)v\,ds + \\ & \sum_{T\in T_h}\int_{\partial T^-\cap\Gamma_j} \beta\cdot n y v\,ds, \end{aligned}$$

and

$$(f_h, v_h) = \sum_{T \in T_h} \int_T \sum_{i=1}^{m} u_i \chi_i v \, dx + \sum_{E \in \varepsilon_h^\delta} \int_E \alpha_j (y - y_a) v \, ds.$$

For time discretization we use the Euler type method. Let N_T be a positive integer. The discrete time interval $I = [0, T]$ is defined as $0 = t_0 < t_1 < \cdots < t_{N_T - 1} < t_{N_T} = T$ with the size $\tau_n = t_n - t_{n-1}$, for $n = 1, \cdots, N_T$, and $\tau = \max_{n=1,\cdots,N_T} \tau_n$.

The discrete approximation scheme of the semi-discrete problem (2) using the backward Euler type method is

$$\min_{u_n \in U_{ad}} \sum_{n=1}^{N_T} \tau_n \left(\frac{w}{2} \sum_{T \in T_h} ||y_{h,n} - y_n^d||^2_{L^2(T)} + \frac{1-w}{2} \sum_{i=0}^{m} (u_{i,n})^2 \right),$$

with

$$\begin{aligned} \left(\frac{y_{h,n} - y_{h,n-1}}{\tau_n}, v_h \right) + a(y_{h,n}, v_h) &= (f, v_h), \quad \forall v_h \in V_h, \\ (y(0,x), v_h) &= (y^0, v_h), \end{aligned}$$

where

$$U_{ad} = \{ u \in L^2(0,T) : u_a \le u_n \le u_b \quad \text{in} \quad (0,T) \}, \quad \text{for} \quad n = 1, 2, ..., N_T.$$

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