An approach for solving a Hamilton -Jacobi type problem with fractional diffusion

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We study in our paper a class of Dirichlet-type boundary value problems that are driven by the fractional Laplacian. More precisely, let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded domain with boundary $\partial\Omega$ of class C^2. For $s \in (0,1)$ and f in a suitable Lebesgue space , we discuss, first the following fractional Poisson problem

$$\begin{cases} (-\Delta)^s u = f, & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1}$$

where $(-\Delta)^s$ is the classical fractional Laplacian defined, for $s \in (0,1)$, by

$$(-\Delta)^s u(x) := a_{N,s} \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_\epsilon(0)} \frac{u(x)-u(y)}{|x-y|^{N+2s}} dy, \tag{2}$$

with

$$a_{N,s} := 2^{2s-1} \pi^{-\frac{N}{2}} \frac{\Gamma(\frac{N+2s}{2})}{|\Gamma(-s)|}.$$

Here, Γ is the Gamma function.

Our primary goal is to obtain a global fractional Calderón-Zygmund regularity theory for problem (1). Our main result in that direction reads as follows

Theorem 1. *Assume that $s \in (0,1)$ and let $f \in L^m(\Omega)$ with $m \geq 1$ and u be the unique solution to the fractional Poisson problem (1). Then, we have*

1. *If $1 \leq m < \frac{N}{s}$, then for all $1 \leq p < \frac{mN}{N-ms}$, there exists a positive constant $C = C(N,s,p,m,\Omega)$ such that*

$$\|(-\Delta)^{\frac{s}{2}} u\|_{L^p(\mathbb{R}^N)} \leq C \|f\|_{L^m(\Omega)}.$$

.

2. *$m \geq \frac{N}{s}$, then for all $1 \leq p < \infty$, there exists a positive constant $C = C(N,s,p,m,\Omega)$ such that*

$$\|(-\Delta)^{\frac{s}{2}} u\|_{L^p(\mathbb{R}^N)} \leq C \|f\|_{L^m(\Omega)}.$$

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Our proofs can be found in [1] and are based on a pointwise estimate of the fractional gradient of the Green's function associated to the fractional Laplacian.

As an application, using a fixed point argument, we obtain existence results for the Hamilton-Jacobi KPZ type problem below

$$\begin{cases} (-\Delta)^s u = \mu(x) |\mathbb{D}_s(u)|^q + \rho f(x), & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{3}$$

where $q > 1$, $\rho > 0$, $\mu \in L^\infty(\Omega)$ and $\mathbb{D}_s$ represents one of the following nonlocal "gradient terms"

- $Grad_1 = (-\Delta)^{\frac{s}{2}} u(x) := a_{N,\frac{s}{2}} \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_\epsilon(0)} \frac{u(x) - u(y)}{|x-y|^{N+s}} dy$ (Half s–Laplacian), (Grad$_1$)
- $Grad_2 = \nabla^s u(x) := \mu_{N,s} \int_{\mathbb{R}^N} \frac{(x-y)(u(x) - u(y))}{|x-y|^{N+s+1}} dy$ (Riesz s–Gradient), (Grad$_2$)
- $Grad_2 = \mathcal{D}_s u(x) := \left(\frac{a_{N,s}}{2} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))^2}{|x-y|^{N+2s}} dy. \right)^{\frac{1}{2}}$ (Stein s–Functional).. (Grad$_3$)

Here

$$a_{N,\sigma} := \frac{2^{2\sigma-1}\Gamma\left(\frac{N}{2} + \sigma\right)}{\pi^{\frac{N}{2}} |\Gamma(-\sigma)|} \qquad \text{and} \qquad \mu_{N,\sigma} := \frac{2^\sigma \Gamma(\frac{N+\sigma+1}{2})}{\pi^{\frac{N}{2}} \Gamma(\frac{1-\sigma}{2})},$$

are normalization constants.

In fact, having at hand statement 1 above we can use the Schauder fixed point theorem to derive our main existence result for problem (3).

Theorem 2. *Let $s \in (0,1)$ and assume that $f \in L^m(\Omega)$ where $m \geq 1$ and $m \geq \frac{2N}{N+2s}$ if $\mathbb{D}_s = \mathcal{D}_s$. Then, there exists $\bar{q} = \bar{q}(m,s)$ such that for all $1 < q < \bar{q}$, problem (3) has a weak solution, for ρ small enough.*

Moreover, the smallness of ρ is proven to be, not only sufficient but also necessary to solve problem (3) in case $\mathbb{D}_s = (-\Delta)^{\frac{s}{2}}$ or $\mathbb{D}_s = \mathcal{D}_s$, according to the following theorems

Theorem 3. *Let $0 < s < 1$, $\mu_2 \geq \mu(x) \geq \mu_1 > 0$, $f \in L^1(\Omega)$ with $f \gneqq 0$ and $q > \frac{2(s+1)}{s+2}$. Then, if $\mathbb{D}_s = (-\Delta)^{\frac{s}{2}}$, (3) has no weak solution for ρ large.*

Theorem 4. *Let $0 < s < 1$, $\mu_2 \geq \mu(x) \geq \mu_1 > 0$, $f \in L^1(\Omega)$ with $f^+ \not\equiv 0$ and $q > 1$. Then, if $\mathbb{D}_s = \mathcal{D}_s$, (3) has no weak solution for ρ large.*

We invite the reader to check [2] for the detailed proofs of Theorems 2, 3 and 4.

1. B. Abdellaoui, A.J. Fernández, T. Leonori, A. Younes, Global fractional Calderón-Zygmund regularity, 2022. https://arxiv.org/pdf/2107.06535.pdf.
2. B. Abdellaoui, A. J. Fernández, T. Leonori, and A. Younes. Deterministic kpz-type equations with nonlocal "gradient terms". Annali di Matematica Pura ed Applicata (1923-), 2023, 202(3), 1451–1468. https://doi.org/10.1007/s10231-022-01288-6 .