Lieb–Thirring Inequalities in Hyperbolic Space

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Consider a Schrödinger operator $-\Delta - V$ on $L^2(\mathbb{R}^n)$, where V is a real-valued function. In their celebrated 1976 paper [1], Lieb and Thirring proved that

$$tr\,(-\Delta - V)_-^\gamma \le L_{\gamma,n} \int_{\mathbb{R}^n} V_+^{\gamma+n/2} dx$$

holds true for finite constants $L_{\gamma,n}$ as long as $\gamma > \max(0, 1-n/2)$. In the case $n = 3, \gamma = 0$ this bound is known as the Cwikel-Lieb-Rozenblum (CLR) inequality and was proven by the three independently from each other, see [2] for instance. The second critical case $n = 1, \gamma = 1/2$ was settled by Weidl in 1996 [3]. The inequality is known to fail for $n = 2, \gamma = 0$. For a comprehensive treatment of the subject, see [4].

We use a lifting argument to extend the Lieb–Thirring inequality to hyperbolic spaces, obtaining

Theorem 1. *Let $\gamma \ge 1/2$ and $n \ge 2$. Then there exists a constant $L_{\gamma,n}$ such that for all real-valued functions Q defined on $\mathbb{H}^n$ for which $Q \in L^{\gamma+n/2}(\mathbb{H}^n, dV)$ holds, the eigenvalues $-\lambda_k = (n-1)^2/4 - \mu_k$ of the operator $-\Delta_{\mathbb{H}^n} - Q$ satisfy*

$$\sum_k \mu_k^\gamma \le L_{\gamma,n} \int_{\mathbb{H}^n} Q_+^{\gamma+n/2} dV,$$

where dV is the Riemannian volume on $\mathbb{H}^n$ and $-\Delta_{\mathbb{H}^n}$ is the Laplace–Beltrami operator.

The talk is based on a forthcoming paper with Ari Laptev.

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4. Frank R., Laptev A., Weidl T. Schrödinger Operators: Eigenvalues and Lieb–Thirring Inequalities. – Cambridge: Cambridge University Press, 2022, 508 p.