On a dynamical system involving Caputo fractional derivative

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Fractional differential equations and differential inclusions have proved to be important tools in modeling many physical and economical phenomena. Recently, considerable attention has been given to the existence of solutions of boundary value problem and boundary conditions for implicit fractional differential equations and integral equations with Caputo fractional derivative.

We are interested in this work in a dynamical system governed by a differential inclusion with subdifferential operators and a differential equation with Caputo fractional derivative. In our development, we proceed by a fixed point method. This investigation is a new contribution in the line of the recent works on this subject, see [1,2,3,4,6].

Notations Let H be a real Hilbert space and let $I := [0, T]$ be an interval of R. Consider a real-valued map $\psi(t, \cdot)$ from H into $[0, +\infty]$ which is proper, lower semi-continuous, convex, and satisfies an assumption expressed in term of its conjugate function $\psi^*(t, \cdot)$.
We denote by $\partial\psi(t, \cdot)$ the subdifferential of $\psi(t, \cdot)$ and by $dom\psi(t, \cdot)$ its domain, for each $t \in I$.
Denote by ${}^cD^\alpha x$ the Caputo fractional derivative of order $\alpha > 0$ of the function x.
Denote by

$$\mathcal{C}^1_H(I) = \{u \in \mathcal{C}_H(I) : \frac{du}{dt} \in \mathcal{C}_H(I)\},$$

where $\frac{du}{dt}$ is the derivative of u,

$$W^{\alpha,\infty}_H(I) = \{u \in \mathcal{C}^1_H(I) : {}^cD^{\alpha-1}u \in \mathcal{C}_H(I);\ {}^cD^\alpha u \in L^\infty_H(I)\},$$

where ${}^cD^{\alpha-1}u$ and ${}^cD^\alpha u$ are the fractional Caputo derivatives of order $\alpha - 1$ and α of u, respectively.

Description of the dynamical system The dynamical system to be studied here is coupled by a differential inclusion governed by a time-dependent subdifferential operator and a fractional differential equation described by

$$-\dot{u}(t) \in \partial\psi(t, u(t)) + f(t, x(t)) \text{ a.e. } t \in I$$

$${}^cD^\alpha x(t) = u(t),\ \ t \in I$$

$$u(0) = u_0 \in dom\psi(0, \cdot)$$

$$x(0) = x_0 \in H,$$

where $f : I \times H \to H$ is a single-valued perturbation satisfying suitable conditions.

Auxiliary results

Recall the following result (see [5,7]).

Proposition 1. *Let ψ satisfy suitable assumptions. Let $h \in L^2_H(I)$ and $x_0 \in dom\psi(T_0, \cdot)$, then the evolution problem*

$$\begin{cases} -\dot{x}(t) \in \partial\psi(t, x(t)) + h(t) & \text{a.e. } t \in I \\ x(0) = x_0 \in dom\psi(0, \cdot), \end{cases}$$

admits a unique absolutely continuous solution $x(\cdot)$ satisfying

$$\|\dot{x}\|^2_{L^2_H(I)} \leq \sigma \|h\|^2_{L^2_{\mathbb{R}}(I)} + \eta, \tag{1}$$

where η and σ are non-negative real constants.

We need the following lemma (see [2]).

Lemma 1. *Let $f \in L^\infty_H(I)$ and $x \in H$. Then, the function $u : I \to H$ is a $W^{\alpha,\infty}_H(I)$-solution to the fractional differential equation*

$$\begin{cases} {}^cD^\alpha u(t) = f(t),\ t \in I \\ u(0) = x, \dot{u}(0) = 0, \end{cases}$$

if and only if

$$u(t) = x + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s) ds,\ t \in I.$$

Main result We are able to prove the following theorem.

Theorem 1. *Let ψ and f satisfy suitable assumptions. Then, for any $(u_0, x_0) \in dom\psi(0,\cdot) \times H$ there are an absolutely continuous map $u : I \to H$, and a $W^{\alpha,\infty}_H(I)$ map $x : I \to H$ solution to the dynamical system above.*

Proof We combine the existence result for the differential inclusion involving time-dependent subdifferential operators in Proposition 1, the result in fractional differential theory in Lemma 1, and the fixed point theorem to establish our main result concerning the dynamical system above.

Future researches. There remain several coupled systems by the evolution problems governed by subdifferential operators with fractional differential equations to be investigated in the line of the large literature on the subject.

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