On error estimates for the approximation of solutions to general boundary-value problems

V. O. Soldatov
Institute of Mathematics, Kyiv, Ukraine
soldatov@imath.kiev.ua, soldatovvo@ukr.net

We present some results on the approximation of solutions to a one-dimensional boundary-value problem of the form

$$Ly(t) \equiv y^{(r)}(t) + \sum_{l=1}^{r} A_{r-l}(t)\, y^{(r-l)}(t) = f(t) \ \text{ for almost all } \ t \in [a,b], \quad By = q. \tag{1}$$

Here, $r, m \in \mathbb{N}$, $a, b \in \mathbb{R}$, $a < b$, each $A_{r-l} \in (L_1)^{m\times m}$, $f \in (L_1)^m$, $q \in \mathbb{C}^{rm}$, and the continuous linear operator $B\colon (C^{(r-1)})^m \to \mathbb{C}^{rm}$ are chosen arbitrarily. The solution y is considered in the Sobolev space $(W_1^r)^m$. This boundary-value problem is called general by analogy with the $r = 1$ case. All function spaces are complex and given on $[a, b]$. We assume that problem (1) has a unique solution $y \in (W_1^r)^m$ for arbitrary $f \in (L_1)^m$ and $q \in \mathbb{C}^{rm}$.

Consider a sequence of general boundary-value problems

$$L_k\, y_k(t) \equiv y_k^{(r)}(t) + \sum_{l=1}^{r} A_{r-l,k}(t)\, y_k^{(r-l)}(t) = f(t) \ \text{ for almost all } \ t \in [a,b], \quad B_k\, y_k = q \tag{2}$$

depending on $k \in \mathbb{N}$ such that, for all $f \in (L_1)^m$ and $q \in \mathbb{C}^{rm}$, each problem (2) has a unique solution $y_k \in (W_1^r)^m$, and $y_k \to y$ in $(W_1^r)^m$ as $k \to \infty$.

Specifically [1, Theorem 1], such a sequence exists and is built explicitly in the class of multi-point boundary-value problems, where each $A_{r-l,k}$ belongs to an arbitrarily chosen dense subset of $(L_1)^{m\times m}$ and

$$B_k\, y_k \equiv \sum_{j=1}^{p_k} \sum_{l=0}^{r-1} \beta_k^{j,l}\, y^{(l)}(t_{k,j}),$$

with all $p_k \in \mathbb{N}$, $\beta_k^{j,l} \in \mathbb{C}^{rm\times m}$, and $t_{k,j} \in [a, b]$.

Suppose that the data of problems (2) depend on k, i.e.

$$L_k\, x_k(t) = f_k(t) \ \text{ for almost all } \ t \in [a,b], \quad B_k\, x_k = q_k, \tag{3}$$

where $f_k \in (L_1)^m$, $q_k \in \mathbb{C}^{rm}$, and $x_k \in (W_1^r)^m$. We present estimates for the approximation error $x_k - y$ in the normed spaces $(W_1^r)^m$ and $(C^{(r-1)})^m$.

Theorem 1. *Let $\varepsilon > 0$ and $\widehat{\varrho} \in \mathbb{N}$. Suppose that*

$$\|f_k - f, (L_1)^m\| < \varepsilon \ \text{ and } \ \|q_k - q, \mathbb{C}^{rm}\| < \varepsilon \ \text{ whenever } \ k \geq \widehat{\varrho}. \tag{4}$$

Then there exist positive numbers $\varkappa$ and $\varrho \geq \widehat{\varrho}$ such that

$$\|x_k - y, (W_1^r)^m\|_{r,1} < \varkappa\, \varepsilon \ \text{ whenever } \ k \geq \varrho.$$

The number $\varkappa$ can be chosen independently of ε, $\widehat{\varrho}$, f, q, f_k, and q_k, whereas the number ϱ can be chosen independently of f_k and q_k.

Let F and F_k denote the primitives of f and f_k on $[a,b]$ subject to $F(a)=0$ and $F_k(a)=0$, resp.

Theorem 2. *Let $\varepsilon>0$ and $\widehat{\varrho}\in\mathbb{N}$. Suppose that*

$$\|F_k-F,(C^{(0)})^m\|<\varepsilon \;\; \textit{and} \;\; \|q_k-q,\mathbb{C}^{rm}\|<\varepsilon \;\; \textit{whenever} \;\; k\geq\widehat{\varrho} \tag{5}$$

and that

$$\sigma:=\sup\bigl\{\|B_k\colon(C^{(r-1)})^m\to\mathbb{C}^{rm}\|\colon k\geq\widehat{\varrho}\bigr\}<\infty.$$

Then there exist positive numbers $\varkappa$ and $\varrho\geq\widehat{\varrho}$ such that

$$\|x_k-y,(C^{(r-1)})^m\|<\varkappa\,\sigma\,\varepsilon \;\; \textit{whenever} \;\; k\geq\varrho. \tag{6}$$

The number $\varkappa$ can be chosen independently of ε, $\widehat{\varrho}$, σ, f, q, and problems (2) *(i.e. $\varkappa$ depends only on L and B), whereas ϱ can be chosen independently of f_k and q_k.*

Comparing these theorems, we note that condition (5) is weaker than (4) and that the norm in $C^{(r-1)}$ is weaker than the norm in W_1^r.

The number $\varkappa$ in (6) can be explicitly indicated. Namely, if $r=1$, we may take

$$\varkappa:=(c_1+c_2)\lambda+c_1c_2+1, \tag{7}$$

where

$$c_1:=1+\|Y,(C^{(0)})^{m\times m}\|\cdot\|[BY]^{-1},\mathbb{C}^{m\times m}\|,$$
$$c_2:=2+\|Y,(C^{(0)})^{m\times m}\|\cdot\|Y^{-1},(C^{(0)})^{m\times m}\|\cdot\|A,(L_1)^{m\times m}\|,$$
$$\lambda:=\|B:(C^{(0)})^m\to\mathbb{C}^m\|^{-1}.$$

Here, Y is the matriciant of the differential system

$$Ly(t)\equiv y'(t)+A(t)y(t)=f(t)$$

related to the point $t=a$, and $[BY]$ is the $m\times m$-matrix whose columns equal the value of B at the corresponding columns of Y. We let the norm of a vector-valued function equal the sum of the norms of its components, and we let the norm of a matrix-valued function equal the maximum of the norms of its columns.

If $r\geq 2$, we may define $\varkappa$ by formula (7), where

$$c_1:=1+\|V,(C^{(0)})^{rm\times rm}\|\cdot\|[BV^{\circ}]^{-1},\mathbb{C}^{rm\times rm}\|,$$
$$c_2:=2+\|V,(C^{(0)})^{rm\times rm}\|\cdot\|V^{-1},(C^{(0)})^{rm\times rm}\|\,(b-a+\|A_{r-1},(L_1)^{m\times m}\|),$$
$$\lambda:=\|B:(C^{(r-1)})^m\to\mathbb{C}^{rm}\|^{-1}.$$

Here, V is the matriciant of the system (1) reduced to first-order system and related to the point $t=a$; V° is the $m\times rm$-matrix function formed by the first m rows of V, and $[BV^{\circ}]$ is defined similarly to $[BY]$.

If each problem (3) is multi-point, these theorems are proved in [1].

1. Murach A. A., Pelekhata O. B., Soldatov V. O. Approximation properties of solutions to multi-point boundary-value problems. Ukrainian Math. J., 2021, **73**, no. 3, 399–413. (arXiv:2012.15604)