Weak Harnack inequality for unbounded minimizers of elliptic functionals with non-standard growth

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In paper [1], we prove the weak Harnack inequality for the functions u which belong to the corresponding De Giorgi classes $DG^{-}(\Omega)$ under the additional assumption that $u \in L^{s}_{loc}(\Omega)$ with some $s > 0$. In particular, our result covers new cases of functionals with a variable exponent or double-phase functionals under the non-logarithmic condition.

Definition 1. We write $W^{1,\Phi(\cdot)}(\Omega)$ for the class of functions $u \in W^{1,1}(\Omega)$ with $\int\limits_{\Omega} \Phi(x,|\nabla u|)dx < \infty$ and we say that a measurable function $u : \Omega \to \mathbb{R}$ belongs to the elliptic class $DG^{\pm}_{\Phi}(\Omega)$ if $u \in W^{1,\Phi(\cdot)}(\Omega)$ and there exist numbers $c > 0$, $q > 1$ such that for any ball $B_{8r}(x_0) \subset \Omega$, any $k \in \mathbb{R}$ and any $\sigma \in (0,1)$ the following inequalities hold:

$$\int\limits_{A^{\pm}_{k,r(1-\sigma)}} \Phi\big(x,|\nabla u|\big)dx \leqslant \frac{c}{\sigma^q} \int\limits_{A^{\pm}_{k,r}} \Phi\left(x,\frac{(u-k)_{\pm}}{r}\right)dx,$$

here $(u-k)_{\pm} := \max\{\pm(u-k),0\}$, $A^{\pm}_{k,r} := B_r(x_0) \cap \{(u-k)_{\pm} > 0\}$.

We suppose that $\Phi(x,v) : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ is a non-negative function satisfying the following properties: for any $x \in \Omega$ the function $v \to \Phi(x,v)$ is increasing and $\lim\limits_{v\to 0} \Phi(x,v) = 0$, $\lim\limits_{v\to+\infty} \Phi(x,v) = +\infty$. We also assume that

(Φ) There exist $1 < p < q$ such that for $x \in \Omega$ and for $w \geqslant v > 0$ there holds

$$\left(\frac{w}{v}\right)^p \leqslant \frac{\Phi(x,w)}{\Phi(x,v)} \leqslant \left(\frac{w}{v}\right)^q.$$

(Φ_λ) There exist $s > 0$, $R > 0$ and continuous, non-decreasing function $\lambda(r) \in (0,1)$ on the interval $(0,R)$, $\lim\limits_{r\to 0} \lambda(r) = 0$, $\lim\limits_{r\to 0} \dfrac{r}{\lambda(r)} = 0$, such that for any $B_r(x_0) \subset B_R(x_0) \subset \Omega$ and some $A > 0$ there holds

$$\Phi^{+}_{B_r(x_0)}\left(\frac{\lambda(r)v}{r^{1+\frac{n}{s}}}\right) \leqslant A\,\Phi^{-}_{B_r(x_0)}\left(\frac{\lambda(r)v}{r^{1+\frac{n}{s}}}\right), \quad r^{1+\frac{n}{s}} \leqslant \lambda(r)v \leqslant 1,$$

here $\Phi^{+}_{B_r(x_0)}(v) := \sup\limits_{x\in B_r(x_0)} \Phi(x,v), \quad \Phi^{-}_{B_r(x_0)}(v) := \inf\limits_{x\in B_r(x_0)} \Phi(x,v), \quad v > 0.$

For the function $\lambda(r)$ we also need the following condition

(λ) For any $0 < r < \rho < R$ there holds

$$\lambda(r) \geqslant \lambda(\rho)\left(\frac{r}{\rho}\right)^b,$$

with some $b \geqslant 0$.

For the function $\lambda(r) = \left[\log \dfrac{1}{r}\right]^{-\frac{\beta}{q-p}}$, $\beta \geqslant 0$ this condition holds evidently, provided that R is small enough.

Remark 1. Consider the function $\Phi(x,v) := v^p + a(x)v^q$, $a(x) \geqslant 0$, $osc_{B_r(x_0)} a(x) \leqslant K r^a \big[\log \frac{1}{r}\big]^{\beta}$, $a \in (0,1]$, $\beta \geqslant 0$, $K > 0$. Evidently condition (Φ_λ) holds with $\dfrac{n(q-p)}{a+p-q} \leqslant s \leqslant \infty$, $a \geqslant q-p$, $\lambda(r) := \big[\log \frac{1}{r}\big]^{-\frac{\beta}{q-p}}$ and $A = K^{q-p}$.

For the function $\Phi(x,v) := v^{p(x)}$, $osc_{B_r(x_0)} p(x) \leqslant \dfrac{L}{\log \frac{1}{r}}$, $L > 0$ condition (Φ_λ) holds with $s > 0$, $\lambda(r) \equiv 1$ and $A = \exp\big(L(1+\frac{n}{s})\big)$.

Our main result reads as follows.

Theorem 1. *Let $u \in DG^-(\Omega)$, $u \geqslant 0$, let conditions (Φ), (Φ_λ), (λ) be fulfilled. Let $B_{8\rho}(x_0) \subset B_R(x_0) \subset \Omega$, let additionally $u \in L^s_{loc}(\Omega)$ with some $s \geqslant q-p$ and $\left(\int\limits_{B_{2\rho}(x_0)} u^s\right)^{\frac{1}{s}} \leqslant d$. Then there exists a positive constant C depending only on the known parameters and d, such that*

$$\left(\frac{1}{|B_\rho(x_0)|}\int\limits_{B_\rho(x_0)} (u+\rho)^\theta dx\right)^{\frac{1}{\theta}} \leqslant \frac{C}{\lambda(\rho)}\left(\inf_{B_{\frac{\rho}{2}}(x_0)} u + \rho\right),$$

where $\theta > 0$ is some fixed number depending only on the known data.

The conditions of the Theorem are precise, we refer the reader to [2] for the examples. In the case $s = \infty$, the Theorem was proved in [3,4].

1. Savchenko M. O., Skrypnik I. I., Yevgenieva Y. A. A note on the weak Harnack inequality for unbounded minimizers of elliptic functionals with generalized Orlicz growth. https://doi.org/10.48550/arXiv.2304.04499
2. Benyaiche A., Harjulehto P., Karppinen P. Hästö, A., The weak Harnack inequality for unbounded supersolutions of equations with generalized Orlicz growth. J. of Diff. Equations, 2021, 275, 790–814.
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