Existence of fixed points in models of dynamic conflict systems with attractive interaction

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Below will construct a model of a dynamic system that describes a uniform or group distribution of individuals population in some resource space of existence $\Omega = \{\omega_1, \omega_2, \dots, \omega_n\}$, $n > 1$. We fix on Ω some probabilistic discrete measure μ, which will correspond to some population. The value of this measure

$$\mu(\omega_i) =: p_i \geq 0, \quad i = 1, \dots, n,$$

characterizes as the density of population distribution in the region of existence ω_i. By construction $\sum\limits_i p_i = 1$. Therefore the sequence $\mu(\omega_i)$ defines in $\mathbb{R}^n_+$ stochastic vector $\mathbf{p} = (p_1, p_2, \dots, p_n)$.

The struggle for vital resources, concentrated in each of the regions of space, is described in terms of measure ν, the value of which at the points ω_i is determined as follows:

$$\nu(\omega_i) = \frac{1 - \mu(\omega_i)}{n-1} = \frac{1 - p_i}{n-1} =: r_i.$$

Obviously, these values also form a stochastic vector $\mathbf{r} = (r_1, r_2, \dots, r_n) \in \mathbb{R}^n_+$. The value $\nu(\omega_i)$ can be interpreted as the averaging of the distribution μ (mean-field) on the subset $\Omega \setminus \omega_i$.

Let's investigate the behavior of the trajectories of a dynamic system with discrete time in terms of stochastic vectors

$$\mathbf{p}^t \xrightarrow{\divideontimes, t} \mathbf{p}^{t+1}, \quad t = 0, 1, \dots \quad (\mathbf{p}^0 \equiv \mathbf{p}), \tag{1}$$

where $\divideontimes$ denotes the transformation that corresponds to the dynamics of the population change in the existence space of Ω.

In terms of coordinates, we define the law of conflict dynamics as follows:

$$p_i^{t+1} = \frac{p_i^t \bigl(1 + c_i r_i^t\bigr)}{z^t}, \quad 0 \leq c_i \leq 1, \quad i = 1, \dots, n, \tag{2}$$

where $r_i^t = \frac{1 - p_i^t}{n-1}$, and the normalizing denominator z^t ensures the stochasticity of the vector $\mathbf{p}^{t+1}$. The set of constants c_i is interpreted as favorable conditions for the existence of the population in each of the regions ω_i, which do not change over time t, respectively.

Note that the works of [1–3] built a mathematical model of a dynamic conflict system that describes the behavior of individuals in society. The dynamics in this model are described by equations with repulsive interaction:

$$p_i^{t+1} = \frac{p_i^t \bigl(1 - c_i r_i^t\bigr)}{z^t}, \quad 0 \leq c_i \leq 1.$$

The choice of attractive interaction fundamentally changes the dynamics of the system and significantly affects its properties. Next, an analysis of the behavior of the trajectories (1), given by equations (2) was performed in terms of coordinates of stochastic vectors $\mathbf{p}^t$, $\mathbf{r}^t$.

Next, we will consider the case when all parameters c_i are different and we will show that it is the set of parameters that completely determines the appearance of the limit state of the system. Due to the fact that a fixed set of constant parameters c_i is set at $t = 0$ and does not depend on time t, the limit state of the system is stable.

Given that $r_i^t = \frac{1-p_i^t}{n-1}$, the equation (2) can be written as

$$p_i^{t+1} = p_i^t \cdot \frac{n-1+c_i(1-p_i^t)}{n-1+L_c^t} = p_i^t \cdot k_{i,c}^t, \quad t \geq 1, \tag{3}$$

where $k_{i,c}^t = \frac{n-1+c_i(1-p_i^t)}{n-1+L_c^t}$, $L_c^t = \sum_{i=1}^n c_i p_i^t(1-p_i^t)$. Let's introduce $\kappa_i^t := c_i^t(1-p_i^t)$.

Theorem 1. *Let $\mathbf{p} \in \mathbb{R}_+^2$. Then each trajectory of the dynamic conflict system* (1) *converges to a stable limit state, which is a fixed point $\mathbf{p}^\infty = \lim_{t\to\infty} \mathbf{p}^t$, moreover $\mathbf{p}^\infty = \left(\frac{c_1}{c_1+c_2}, \frac{c_2}{c_1+c_2}\right)$.*

Theorem 2. *Let $\mathbf{p} \in \mathbb{R}_+^3$, and for each $i = 1, 2, 3$ one of the conditions $\kappa_i^t > L_c^t$ or $\kappa_i^t < L_c^t$ is fulfilled for any t. Then each trajectory of the dynamic conflict system* (1) *converges to the limit state, which is a fixed point $\mathbf{p}^\infty = \lim_{t\to\infty} \mathbf{p}^t$. Moreover,*

$$\mathbf{p}^\infty = \left(\frac{c_1c_2 + c_1c_3 - c_2c_3}{c_1c_2 + c_1c_3 + c_2c_3}, \frac{c_1c_2 - c_1c_3 + c_2c_3}{c_1c_2 + c_1c_3 + c_2c_3}, \frac{c_1c_3 - c_1c_2 + c_2c_3}{c_1c_2 + c_1c_3 + c_2c_3}\right),$$

if $c_i > \frac{c_jc_k}{c_j+c_k}$. for all $i, j, k = 1, 2, 3$, $i \neq j \neq k$

$$c_i > \frac{c_jc_k}{c_j + c_k}. \tag{4}$$

Theorems 1 and 2 assume generalizations to the case $\mathbf{p} \in \mathbb{R}_+^n$, $n > 3$. At the same time, the explicit form of the condition 4 becomes much more complicated and is established for each case $n > 3$ separately.

Theorem 3. *Let $\mathbf{p} \in \mathbb{R}_+^n$, $n > 3$. Suppose that for each $i = 1, \dots, n$ starting from a certain moment in time t^* one of the conditions $\kappa_i^{t^*} > L_c^{t^*}$ or $\kappa_i^{t^*} < L_c^{t^*}$ is fulfilled. Then each trajectory is dynamic of the conflict system* (1)*, збігається до converges to the limit state, which is a fixed point $\mathbf{p}^\infty = \lim_{t\to\infty} \mathbf{p}^t$. Moreover, for each $i = 1, \dots, n$ holds one of the equalities*

$$p_i^\infty = 0 \quad or \quad p_i^\infty = f_i(c_1, c_2, \dots, c_n),$$

where $f_i(c_1, c_2, \dots, c_n)$ is positive.

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