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Turing's instability in the generalized Degn-Harrison system with superdiffusion

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This paper examines the turing instability in a general reaction-diffusion system using the Degn-Harrison model with superdiffusion. A condition of homogeneous stability becomes unstable in the presence of superdiffusion. By analyzing the system's equilibrium point stability, sufficient conditions are established for the existence of Turing instability and Hopf bifurcation.

This article is concerned with a D-H model with a fractional Laplacian. Let's begin by defining the anomalously diffusive operator $(-\Delta)^{\frac{\gamma}{2}}$, which, according to Samko1993 [1],Gorenflo1998 [2], is defined by the Fourier transform as

$$F\left(-(-\Delta)^{\frac{\gamma}{2}}u\right)(\xi) = -|\xi|^{\gamma}Fu(\xi), \qquad \xi \in \mathbb{R}^n, \gamma > 0.$$

Note that $0 < \gamma < 1$ is frequently referred to as subdiffusion, whereas $1 < \gamma < 2$ is referred to as superdiffusion. The operator $(-\Delta)^{\frac{\gamma}{2}}$ becomes the standard diffusion operator (Laplacian operator) when $\gamma = 2$. The aberrant diffusion operator$-(-\Delta)^{\frac{\gamma}{2}}$ is typically denoted by ∇^{γ}. Accordingly, we may write

$$F\left(\nabla^{\gamma}u\right)(\xi) = -|\xi|^{\gamma}Fu(\xi), \qquad \xi \in \mathbb{R}^n, \gamma > 0.$$

The generalized Degn–Harrison reaction–diffusion system proposed in [3]

$$\begin{cases} u_t = d_1\nabla^{\gamma}u + a - \mu u - \lambda\varphi(u)v = f\ (u,v) \\ v_t = d_2\nabla^{\gamma}v + \sigma b\left(u - \varphi(u)v\right) = g(u,v) \end{cases} \tag{1}$$

It is supposed that the constants $d_1, d_2, a, \lambda, \mu$, and σ are positive. It is supposed that the function φ is nonnegative and continuously differentiable on $\mathbb{R}^+$ such that

$$\varphi(0) = 0,$$

and for $u \in]\delta, a[$

$$\varphi(u) > 0$$

and

$$\varphi(u) > 0$$

with

$$0 < \delta < a - b$$

where $\Gamma(.)$ represents the function Gamma. For operators of high dimension, the function u is replaced by $u(x_1, x_2, ..., x_n, t)$, and the operator ∇^{γ} is defined by its Fourier transform, $|\xi| = \sqrt{\xi_1^2 + \xi_2^2 + ... + \xi_n^2}$. The system beginning with (x, y) belonging to the domain $-\pi < x, y < \pi$

in $\mathbb{R}^2$ is subject to a number of conditions. The system is assumed to have initial conditions $u(x,y,0) = u_0(x,y), v(x,y,0) = v_0(x,y)$ and Neumann boundary conditions of type

$$\begin{cases} u_{x,y}(\pm\pi, t) = 0, \\ v_{x,y}(\pm\pi, t) = 0. \end{cases} \tag{2}$$

Additionally, we allow

$$\begin{cases} u(x+2l\pi, y+2m\pi, t) = u(x,y,t), \\ v(x+2l\pi, y+2m\pi, t) = v(x,y,t), \end{cases} \tag{3}$$

and suppose

$$h = \sqrt{h_1^2 + h_2^2}, \tag{4}$$

where $(h_1, h_2) \in \mathbb{R}^2$ exists. Periodic replication is used to define the function u, v on $\mathbb{R}^2$.

Theorem 1. *(i) The system (2) undergoes Hopf bifurcation at $k = k_H$ in the absence of superdiffusion. If $k < k_H$ is , then the point (u_0, v_0) is unstable; if $k \geq k_H$, then the point is asymptotically stable.*

(ii) If conditions $k_c < k < \bar{k}$, or $b_H < k < k_c$ or $[(x_+)^{\frac{1}{\gamma}} - (x_-)^{\frac{1}{\gamma}}] < 1$ are satisfied under conditions $d_2 > d_1$ and (H_1), then no Turing instability could occur in the system (2);

(iii) Under $d_2 > d_1$ and (H_1) ,if $k_H < k < k_c$ and $[(x_+)^{\frac{1}{\gamma}} - (x_-)^{\frac{1}{\gamma}}] \geq 1$ hold, the system will experience Turing instability. (2).The minimal value of D_h in relation to $|h|^\gamma$ at (h_c, k_c) is negative.

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