International Conference of Young Mathematicians
The Institute of Mathematics of the National Academy of Sciences of Ukraine
June 1–3, 2023, Kyiv, Ukraine

Dynamic integral inequalities with their applications in terminal value problems on time scales

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The evolution of the new mathematical inequalities both continuous and discrete often places a rigid support for the interrogative procedures and algorithms practice in applied sciences. It is ordinary to ask whether it is plausible to have a scheme which incorporate both discrete and continuous structures simultaneously. In order to combine discrete and continuous analysis, Stefen Hilger successfully introduced the notion of time scale calculus [1, 2]. Since then hundreds of research articles came out in this theory and its applications to several fields, see e.g., the very striking treatises of Bohner and Peterson in [3, 4].

The theory of dynamic equations has been evolve very intensively in the last several decades, see [4, 5]. These equations have a very special transition property to differential and difference equations respective to the time scales of real and integer numbers. The theory of the boundary value problem for dynamic equations is quite interesting field in recent era of mathematics. In this regard, few manual are dedicated see [6, 7, 8]. However the study of terminal value problem (TVP) for dynamic equations is varsatile and emerging area in the theory of dynamic non-linear equation. In [9], "Hilsher and C.C Tisdell" discuss TVP of order first and second. This assertion contain the qualitative analysis of certain differential equations which are partial in nature and this analysis includes uniqueness and boundedness of their solution.

In this article we develop new dynamic integral inequalities on an arbitrary time scale. These inequalities gives us the bound on the unknown functions which is right dense continuous on any arbitrary time scale. The integral inequalities in their more general forms have unknown functions inside the integral as well so claiming about the maximum value about these unknown functions is not straight forward. Our results provides a promising solution for this caveat in such a way that our bounds provide a explicit bounds for these unknowns. Although these bounds are not for any arbitrary functions, but they are applicable to a much larger class of functions i.e right dense continuous functions. A glimpse of one of these results is:

Theorem 1. *Let $E(\breve{\imath},\breve{s}), E_1(\breve{\imath},\breve{s}), E_2(\breve{\imath},\breve{s}), E_3(\breve{\imath},\breve{s})$ belongs to $\mathrm{C}_{rd}(\Lambda^2,\mathbb{R}^+)$ which are defined for all $\breve{\imath},\breve{s}\in\mathbb{T}_0$.*
(a_1) *If*

$$E(\breve{\imath},\breve{s}) \le E_1(\breve{\imath},\breve{s}) + E_2(\breve{\imath},\breve{s}) \int_{\breve{\imath}_0}^{\breve{\imath}} \int_{\breve{s}}^{\infty} E_3(s,t)E(s,t)\Delta t\Delta s \tag{1}$$

for $\breve{\imath},\breve{s}\in\mathbb{T}_0$, then

$$E(\breve{\imath},\breve{s}) \le E_1(\breve{\imath},\breve{s}) + E_2(\breve{\imath},\breve{s})A(\breve{\imath},\breve{s})e_{\kappa_1}(\breve{\imath},\breve{\imath}_0) \tag{2}$$

where

$$A(\breve{\imath},\breve{s}) = \int_{\breve{\imath}_0}^{\breve{\imath}} \int_{\breve{s}}^{\infty} E_3(s,t)E_1(s,t)\Delta t\Delta s \tag{3}$$

and $\kappa_1 = \int_{\breve{s}}^{\infty} E_3(\breve{\imath},t)E_2(\breve{\imath},t)\Delta t$.
(a_2) *If*

$$E(\breve{\imath},\breve{s}) \le E_1(\breve{\imath},\breve{s}) + E_2(\breve{\imath},\breve{s}) \int_{\breve{\imath}}^{\infty} \int_{\breve{s}}^{\infty} E_3(s,t)E(s,t)\Delta t\Delta s \tag{4}$$

for $\breve{\iota}, \breve{s} \in \mathbb{T}_0$, *then*

$$E(\breve{\iota}, \breve{s}) \leq E_1(\breve{\iota}, \breve{s}) + E_2(\breve{\iota}, \breve{s})\bar{A}(\breve{\iota}, \breve{s}))e_{\kappa_1(\infty,\breve{\iota})} \tag{5}$$

where

$$\bar{A}(\breve{\iota}, \breve{s}) = \int_{\breve{\iota}}^{\infty}\int_{\breve{s}}^{\infty} E_3(s,t)E_1(s,t)\Delta t\Delta s \tag{6}$$

and $\kappa_1 = \int_{\breve{s}}^{\infty} E_3(\breve{\iota}, t)E_2(\breve{\iota}, t)\Delta t$.

Theses types of inequalities present in the literature for the case of real and integers domain. Our results are more general in the sense that they are applicable to every closed subset of real numbers and if we take our time scale (i.e the closed subset of real numbers) as integers and real numbers our results transformed into special cases already present in the literature [10] and validate these results.

The work is not merely the work of given bounds to inequalities and generalize them to any subset of real numbers but we also provide how we use these bounds to solve particular type differential and difference equations (more generally dynamic equations). The results are applicable for the qualitative analysis such as uniqueness and boundness of the solution of terminal value problems for certain partial dynamic equation given below:

$$\begin{gathered} E^{\Delta_1\Delta_2}(\breve{\iota}, \breve{s}) = h(\breve{\iota}, \breve{s}, E^{\sigma}(\breve{\iota}, \breve{s})) + r(\breve{\iota}, \breve{s}) \\ E(\breve{\iota}, \infty) = \sigma_\infty(\breve{\iota}), E(\infty, \breve{s}) = \tau_\infty(\breve{s}), E(\infty, \infty) = d \end{gathered} \tag{7}$$

where the functions $h : \Lambda^2 \times \mathbb{R} \to \mathbb{R}, r : \Lambda^2 \to \mathbb{R}, \sigma_\infty, \tau_\infty : \mathbb{T}_0 \to \mathbb{R}$ are all right dense continuous functions and d is a real constant.

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