An abstract second order differential equation with Robin boundary conditions in UMD spaces

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In this paper we study an abstract second order differential equation of elliptic type with variable operator coefficients and general Robin boundary conditions, in the framework of UMD spaces. These problems presents for example the linearized stationary case of a model describing information diffusion in online social networks. Existence and regularity results are obtained when the Labbas-Terreni assumption is fulfilled using semi-groups theory and interpolation spaces.

This paper is devoted to study the following general problem

$$\begin{cases} u''(x) + A(x)\, u(x) - \omega u(x) = f(x), \qquad x \in (0,1) \\ u'(0) - Hu(0) = d_0 \\ u(1) = u_1, \end{cases} \tag{1}$$

with $f \in L^p(0,1;E)$, $1 < p < +\infty$, where E is a complex Banach space, d_0, u_1 are given elements in E and $(A(x))_{x\in[0,1]}$ is a family of closed linear operators whose domains $D(A(x))$ are dense in E. H is a closed linear operator in E, ω is a positive real number. The results proved here in the L^p case complete our recent paper concerning the hölderian case, see [2].

For all $x \in [0,1]$, set $A_\omega(x) = A(x) - \omega I$.

We will seek for a classical solution u to (1), i.e. a function u such that

$$\begin{cases} \text{a.e } x \in (0,1), \quad u(x) \in D(A(x)) \text{ and} \\ x \mapsto A(x)\, u(x) \in L^p(0,1;E) \\ u \in W^{2,p}(0,1;E) \\ u(0) \in D(H), \end{cases} \tag{2}$$

The method is essentially based on Dunford calculus, interpolation spaces, the semigroup theory and some techniques as in [2, 3].

We will assume that

$$E \ \text{ is a } \ UMD \ \text{ space} \tag{3}$$

and suppose that

$$\exists\, \omega_0 > 0,\ \exists C > 0 : \forall x \in [0,1],\ \forall z \geq 0,\ (A_{\omega_0}(x) - zI)^{-1} \in L(E)$$

and

$$\left\|(A_{\omega_0}(x) - zI)^{-1}\right\|_{L(E)} \leq \frac{C}{1+z}; \tag{4}$$

and setting $Q_\omega(x) = -(-A_\omega(x))^{1/2}$ (see [1]), we suppose also that: $\exists\ C,\ \alpha,\ \mu > 0 : \forall x$, $\tau \in [0,1]$, $\forall\ \omega \geq \omega_0$:

$$\begin{cases} \left\|Q_\omega(x)\,(Q_\omega(x) - zI)^{-1}\left(Q_\omega(x)^{-1} - Q_\omega(\tau)^{-1}\right)\right\|_{L(E)} \leq \dfrac{C\,|x-\tau|^\alpha}{|z+\omega|^\mu} \\ \text{with } \alpha + \mu - 2 > 0; \end{cases} \tag{5}$$

this hypothesis is well known as Labbas-Terreni assumption.

We obtain the following theorem.

Theorem 1. *Assume* (3)–(5)*. Let* $f \in L^p(0,1;E)$*,* $1 < p < +\infty$ *and*

$$\left(Q_\omega(0) - H\right)^{-1} d_0 \in (D(A(0)),E)_{\frac{1}{2p},p}, \quad u_1 \in (D(A(1)),E)_{\frac{1}{2p},p}.$$

Then there exists $\omega^* > 0$ *such that for all* $\omega \geq \omega^*$*, the problem (1) has a unique solution* $w(\cdot) = Q_\omega(\cdot)^2 u(\cdot)$ *verifying*

1. $Q_\omega(\cdot)^2 u(\cdot) \in L^p(0,1;E)$.
2. $u'' \in W^{2,p}(0,1;E)$.

1. Balakrishnan A. V. Fractional Powers of Closed Operators and the Semigroups Generated by them. Pacific J. Math., 1960, 10, 419–437.
2. Haoua R., Medeghri A. Robin boundary value problems for elliptic operational differential equations with variable operators. Electronic Journal of Differential Equations, 2015, 2015, No. 87, 1–19.
3. Labbas R. Problèmes aux limites pour une equation différentielle abstraite de type elliptique, Thèse d'état, Université de Nice, 1987.