Weak chaos in discrete systems

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For a general discrete dynamics on a Banach and Hilbert spaces we give a necessary and sufficient conditions of the existence of bounded solutions under assumption that the homogeneous difference equation admits an discrete dichotomy on the semi-axes. We consider the so called resonance (critical) case when the uniqueness of solution is disturbed. We show that admissibility can be reformulated in the terms of generalized or pseudoinvertibility. As a corollary of the main result we obtain the conditions of weak homoclinic chaos.

Consider the following weakly nonlinear boundary-value problem

$$x_{n+1}(\varepsilon) = A_n x_n(\varepsilon) + \varepsilon Z(x_n(\varepsilon), n, \varepsilon) + h_n, \tag{1}$$

$$lx.(\varepsilon) = \alpha \tag{2}$$

in the Hilbert space $\mathcal{H}, \mathcal{H}_1$ where $A_n : \mathcal{H} \to \mathcal{H}$ - is a set of bounded operators, from the Hilbert space $\mathcal{H}$ into itself. Assume that

$$A = (A_n)_{n \in \mathbb{Z}} \in l_\infty(\mathbb{Z}, \mathcal{L}(\mathcal{H})), \quad h = (h_n) \in l_\infty(\mathbb{Z}, \mathcal{H}).$$

$l : l_\infty(\mathbb{Z}, \mathcal{H}) \to \mathcal{H}_1$ is a linear and bounded operator which translates bounded solutions of (1) into the Hilbert space $\mathcal{H}_1$, α is an element of the Hilbert space $\mathcal{H}_1$. The nonlinear vector-valued function $Z(x(n, \varepsilon), n, \varepsilon)$ satisfies the following conditions

$$Z(\cdot, n, \varepsilon) \in C[||x - x^0|| \le q], Z(x(n, \varepsilon), \cdot, \varepsilon) \in l_\infty(\mathbb{Z}, \mathcal{H}), Z(x(n, \varepsilon), n, \cdot) \in C[0, \varepsilon_0]$$

in the neighborhood of solution $x_n^0(c)$ of the generating ($\varepsilon = 0$) linear problem (q is a small enough constant)

$$x_{n+1} = A_n x_n + h_n, \tag{3}$$

$$lx. = \alpha, \tag{4}$$

We are looking for necessary and sufficient conditions for the existence of strong generalized solutions $x_n(\varepsilon) : \mathbb{Z} \to \mathcal{H}$ of (1), (2) bounded on the entire integer axis

$$x.(\varepsilon) \in l_\infty(\mathbb{Z}, \mathcal{H}), \quad x_n(\cdot) \in C[0, \varepsilon_0],$$

which turn into one of the strong generalized solutions $x_n^0(c)$ of the generating boundary-value problem (1), (2) for $\varepsilon = 0$: $x_n(0) = x_n^0(c)$.

Theorem 1. *Suppose that the homogeneous equation admits an exponential dichotomy on the semi-axes $\mathbb{Z}_+, \mathbb{Z}_-$ with projectors P and Q respectively ($D = P - I + Q$)) and the following condition*

$$\sum_{k=-\infty}^{+\infty} \overline{H}(k+1) h_k = 0 \quad (\overline{H}(n+1) = P_{\overline{\mathcal{H}_D}} Q U^{-1}(n+1), \quad P_{\overline{\mathcal{H}_D}} = I - \overline{DD^+})$$

is satisfied ($U(n)$ is an evolution operator of the homogeneous equation).

Under condition

$$P_{\overline{V}_{\mathcal{H}_1}}(\alpha - l(G[h])(\cdot)) = 0, \quad (V = lU(\cdot))PP_{N(D)} : \mathcal{H} \to \mathcal{H}_1)$$

boundary-value problem (3), (4) has a set of strong generalized solutions in the form

$$x_n^0(\overline{c}) = U(n)PP_{N(D)}P_{N(V)}\overline{c} + \overline{G[h,\alpha]}(n), \quad \overline{c} \in \mathcal{H}$$

where

$$\overline{G[h,\alpha]}(n) = (G[h](n)) + \overline{V}^+(\alpha - l(G[h])(\cdot))$$

is the extension of the generalized Green's operator.

Theorem 2 (necessary condition)**.** *Suppose that the homogeneous equation admits a dichotomy on the semi-axes $\mathbb{Z}_+$ and $\mathbb{Z}_-$ with projectors P and Q respectively. Let the boundary-value problem (1), (2) has a strong generalized solution $x_n(\varepsilon)$ bounded on $\mathbb{Z}$, which turns into one of the generating solutions $x_n^0(c)$ of the boundary-value problem (3), (4) with element $c = c^* \in \overline{\mathcal{H}}$. Then the element c^* satisfies the equation*

$$F(c^*) = \begin{cases} \sum_{k=-\infty}^{+\infty} \overline{H}(k+1)Z(U(k)PP_{N(D)}P_{N(V)}c^* + \overline{(G[h,\alpha])}(k), k, 0) = 0, \\ P_{\overline{V}_{\mathcal{H}_1}} lZ(U(\cdot)PP_{N(D)}P_{N(V)}c^* + \overline{(G[h,\alpha])}(\cdot), \cdot, 0) = 0. \end{cases}, \tag{5}$$

Theorem 3 (sufficient condition)**.** *Suppose that the homogeneous equation admits a dichotomy on the semi-axes $\mathbb{Z}_+, \mathbb{Z}_-$ with projectors P and Q respectively and the considered linear boundary-value problem (3), (4) has strong generalized bounded solutions $x_n^0(c)$. Assume that*

$$P_{\overline{\mathcal{H}}_{\overline{B}_0}} \begin{bmatrix} P_{\overline{\mathcal{H}}_{\overline{D}}} Q \\ P_{\overline{V}_{\mathcal{H}_1}} \end{bmatrix} = 0. \tag{6}$$

Then for each element $c = c^*$ satisfying the equation for generating elements (5) there are strong generalized solutions $x_n(\varepsilon)$ of the nonlinear boundary-value problem (1), (2) bounded on the entire $\mathbb{Z}$ axis, turn for $\varepsilon = 0$ into the generating solutions $x_n^0(c^*) : x_n(0) = x_n^0(c^*)$. These solutions can be found using a convergent iterative process for $\varepsilon \in [0, \varepsilon_*] \subset [0, \varepsilon_0]$

$$y_n^{l+1} = U(n)PP_{N(D)}P_{N(V)}\overline{c}^{l+1}(\varepsilon) + \overline{y}_n^{l+1}(\varepsilon),$$

$$\overline{c}^{l+1}(\varepsilon) = -\overline{B}_0^+ \begin{bmatrix} \sum_{k=-\infty}^{+\infty} \overline{H}(k+1)\left(A_1(k)\overline{y}_k^{l+1}(\varepsilon) + \mathcal{R}(y_k^l(\varepsilon), k, \varepsilon)\right) \\ P_{\overline{V}_{\mathcal{H}_1}} l\left(A_1(\cdot)\overline{y}_\cdot^{l+1}(\varepsilon) + \mathcal{R}(y_\cdot^l(\varepsilon), \cdot, \varepsilon)\right) \end{bmatrix} + \mathcal{P}_{N(B_0)}c_\rho(\varepsilon),$$

$$\overline{y}_n(\varepsilon) = \varepsilon\overline{G[Z(y_\cdot(\varepsilon) + x_\cdot^0(c^*)), 0]}(n),$$

$$x_n^l(\varepsilon) = y_n^l(\varepsilon) + x_n^0(c^*), \quad y_n^0(\varepsilon) = 0, \quad l = \overline{0, \infty}.$$

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