Fractional Differential Perturbed Equations with Infinite State-Dependent Delay in Fréchet Spaces

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In this work we investigate the existence of solutions of some classes of perturbed partial functional hyperbolic differential equations with infinite state-dependent delay involving the Caputo fractional derivative by using a nonlinear alternative of Avramescu in Fréchet spaces.

This work deals with the existence and of solutions to fractional order initial value problem (IVP for short), for the system

$$({}^cD_0^r u)(t,x) = f(t,x,u_{(\rho_1(t,x,u_{(t,x)}),\rho_2(t,x,u_{(t,x)}))})$$
$$+g(t,x,u_{(\rho_1(t,x,u_{(t,x)}),\rho_2(t,x,u_{(t,x)}))}), \text{ if } (t,x)\in J, \tag{1}$$

$$u(t,x) = \phi(t,x), \text{ if } (t,x)\in \tilde{J}', \tag{2}$$

$$\begin{cases} u(t,0)=\varphi(t), \\ u(0,x)=\psi(x), \end{cases} \quad (t,x)\in J, \tag{3}$$

where $\tilde{J}' := (-\infty,+\infty)\times(-\infty,+\infty)\backslash[0,\infty)\times[0,\infty), \varphi,\psi : [0,\infty)\to\mathbb{R}^n$ are given absolutely continuous functions, $f,g : J\times\mathcal{B}\to \mathbb{R}^n,\ \rho_1,\rho_2 : J\times\mathcal{B}\to(-\infty,+\infty)$ are given functions, $\phi : \tilde{J}'\to\mathbb{R}^n$ is a given continuous function with $\phi(t,0)=\varphi(t),\ \phi(0,x)=\psi(x)$ for each $(t,x)\in J$ and $\mathcal{B}$ is called a phase space.

Below we introduce notations, definitions, and preliminary facts which are used throughout this paper.

By $L^1(J,\mathbb{R}^n)$ we denote the space of Lebesgue-integrable functions $u : J\to\mathbb{R}^n$ with the norm

$$\|u\|_{L^1} = \int_0^a\int_0^b \|u(t,x)\|dxdt,$$

where $\|\cdot\|$ denotes a suitable complete norm on $\mathbb{R}^n$.

Definition 1. Let $r=(r_1,r_2)\in(0,\infty)\times(0,\infty), \theta=(0,0)$ and $u\in L^1(J,\mathbb{R}^n)$. The left-sided mixed Riemann-Liouville integral of order r of u is defined by

$$(I_\theta^r u)(t,x) = \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}u(s,\tau)d\tau ds.$$

Definition 2. Let $r\in(0,1]\times(0,1]$ and $u\in L^1(J,\mathbb{R}^n)$. The mixed fractional Riemann-Liouville derivative of order r of u is defined by the expression

$$D_\theta^r u(t,x) = (D_{tx}^2 I_\theta^{1-r}u)(t,x)$$

and the Caputo fractional-order derivative of order r of u is defined by the expression

$$({}^cD_0^r u)(t,x) = \left(I_\theta^{1-r}\frac{\partial^2}{\partial t\partial x}u\right)(t,x).$$

In the sequel we will make use of the following generalization of Gronwall's lemma for two independent variables and singular kernel.

Lemma 1. *Let* $\upsilon : J \to [0, \infty)$ *be a real function and* $\omega(\cdot, \cdot)$ *be a nonnegative, locally integrable function on* J*. If there are constants* $c > 0$ *and* $0 < r_1, r_2 < 1$ *such that*

$$\upsilon(t,x) \leq \omega(t,x) + c\int_0^t \int_0^x \frac{\upsilon(s,\tau)}{(t-s)^{r_1}(x-\tau)^{r_2}} d\tau ds,$$

then there exists a constant $\delta = \delta(r_1, r_2)$ *such that*

$$\upsilon(t,x) \leq \omega(t,x) + \delta c\int_0^t \int_0^x \frac{\omega(s,\tau)}{(t-s)^{r_1}(x-\tau)^{r_2}} d\tau ds,$$

for every $(t,x) \in J$*.*

Theorem 1 (Nonlinear Alternative of Avramescu)**.** *Let* $(X, |\cdot|_n)$ *be a Fréchet space and let* $A, B : X \to X$ *two operators. Suppose that the following hypothesis are fulfilled:*

(*i*) A *is a compact operator;*

(*ii*) B *is a contraction operator with respect to a family of seminorms* $||\cdot||_n$ *equivalent with the family* $|\cdot|_n$*;*

(*iii*) *the set* $\mathcal{E} = \{u \in X : u = \lambda A(u) + \lambda B(\frac{u}{\lambda})$ *for some* $\lambda \in (0,1)\}$ *is bounded.*

Then there is $u \in X$ *such that* $u = Au + Bu$*.*

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