A TYPE OF THE EVOLUTION FREE BOUNDARY PROBLEMS IN A BOUNDED DOMAIN OF $\mathbb{R}^3$: STUDY THE UNIQUENESS OF THE SOLUTION

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We present the uniqueness of the solution of a nonlinear evolution dam problem in the case of an incompressible fluid and Ω is a heterogeneous meduim bounded in $\mathbb{R}^3$. Let Ω be a bounded heterogeneous medium of $\mathbb{R}^3$ with a horizontal background and $y = (y_1, y_2, y_3)$, we set $y = (y', y_3)$ where $y' = (y_1, y_2)$. Let A, B, D and F be real numbers such that $B > A$ and $F > E$. The impermeable part is $\Gamma_1 = [A, B] \times [E, F]$ and Γ_2 is the permeable part. We also give the following definitions

$$\Sigma_1 = \Gamma_1 \times (0, T), \quad \Sigma_2 = \Gamma_2 \times (0, T), \quad \Sigma_3 = \Sigma_2 \cap \{\varphi > 0\}, \quad \Sigma_4 = \Sigma_2 \cap \{\varphi = 0\},$$

where φ is a definite positive Lipschitz function on $\bar{Q}$ of class $C^{0,1}$ in y and from C^1 at t which represents the pressure assigned to Γ_2. The matrix permeability of the porous medium is given by

$$M(y) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & k(y') \end{pmatrix},$$

where $k : (A, B) \times (E, F) \longrightarrow \mathbb{R}^2$ is a function of the variable y'. Then the fluid velocity is

$$v = (k(y'))^{p-1} |u_{y_3}|^{p-2} u_{y_3}.$$

If we set $h(y') = (k(y'))^{p-1}$ and $b(y, u_{y_3}) = |u_{y_3}|^{p-2} u_{y_2}$, we can rewrite v as follows

$$v = h(y') b(y, u_{y_3}).$$

Consider the following weak formulation of the nonlinear evolution dam problem associated with initial data g_0

$$(P) \qquad \begin{cases} \text{Find } (u, g) \in L^p(0, T, W^{1,p}(\Omega)) \times L^\infty(Q) \text{ such that :} \\ u \geq y_3, \quad 0 \leq g \leq 1, \quad g(u - y_3) = 0 \text{ a.e. in Q}, \\ u = \phi \text{ on } \Sigma_2, \\ \displaystyle\int_Q \Big[h(y')(a(y, u_{y_3}) - g a(y, 1)) \xi_{y_2} + g \xi_t \Big] dy dt + \int_\Omega g_0(y) \xi(y, 0) dy \leq 0, \\ \forall\, \xi \in W^{1,q}(Q), \quad \xi = 0 \text{ on } \Sigma_3, \quad \xi \geq 0 \text{ on } \Sigma_4, \\ \xi(x, T) = 0 \text{ for a.e. } y \in \Omega. \end{cases}$$

where $h : (A, B) \times (E, F) \longrightarrow \mathbb{R}^2$ is a Lipschitz continuous function of the variable y' such that for two positive constants m_1 and m_2

$$m_1 \leqslant h(y') \leqslant m_2, \quad \forall\, y' \in (A, B) \times (E, F), \tag{1}$$

$a : \Omega \times \mathbb{R}^3 \longrightarrow \mathbb{R}$ is a continuous function satisfying for some positive constants α, β,

$$\forall\, r \in \mathbb{R} : \quad \alpha\, |r|^p \leqslant a(y, r) r, \tag{2}$$

$$\forall\, r \in \mathbb{R}^3 : \quad |a(y,r)| \leqslant \beta\, |r|^{p-1}\,, \tag{3}$$

$$\forall\, r_1, r_2 \in (\mathbb{R}^3)^2, r_1 \neq r_2 : \quad (a(y,r_1) - a(y,r_2))(r_1 - r_2) > 0, \tag{4}$$

and $g_0 : \Omega \longrightarrow \mathbb{R}$ is a measurable function that satisfies

$$0 \leqslant g_0 \leqslant 1 \;\; \text{a.e. in} \;\; \Omega. \tag{5}$$

We use Thychonoff's fixed point theorem and a methode similar in [1] to prove the existence of the solution to our problem. In [2, 3] using the method of doubling variables, the authors prove the uniqueness of solution of a evolution dam problem, we use a methode similar to prove the following theorem

Theorem 1. *Assume that* $(1) - (5)$ *and* $(0, h(y')a(y,1))\nu \leq 0$ *on* Γ_1 *hold, where* ν *denotes the outward unit normal to the boundary* $\partial\Omega$*. Then, the solution of the problem (P) associated with the initial data* g_0 *is unique.*

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2. Lyaghfouri A., The evolution dam problem for nonlinear Darcy's law and Dirichlet boundary conditions. Port. Math., 1999, 56, No. 1, 1–38.
3. Zaouche E., Uniqueness of solution of a heterogeneous evolution dam problem associated with a compressible fluid flow through a rectangular porous medium. Glas. Mat. Ser. III, 2022, 55, 93–99.