Limit cycles of an integrable class of quintic differential systems with a degenerate singular point

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Among the most important topics arising in the qualitative theory of differential systems, limit cycles and first integrals usually occur. Limit cycles are isolated periodic solutions which occur particularly for autonomous planar differential systems. For more details, the reader can see [1-3], and the references therein. Many authors have studied the existence of one or more limit cycles for quintic differential systems with non-null linear parts, see [4-6].
In this paper, our goal is to study the singular points, the integrability and the existence of limit cycles for a class of quintic differential systems of the form

$$\begin{cases} \dot{x} = P(x,y) = cx^3 + cxy^2 + ax^5 + 2(b-2)x^2y^3 + 2(b+1)x^4y + axy^4 - 2y^5, \\ \dot{y} = Q(x,y) = cx^2y + cy^3 + 2x^5 + ax^4y + 2(b+2)x^3y^2 + 2(b-1)xy^4 + ay^5, \end{cases} \tag{1}$$

where a, b and c are real constants, such that $c \neq 0$ and the dot over the letters denotes derivative with respect to the time t.

More precisely, we prove that this class has a degenerate singular point at the origin and no singularities at infinity. Furthermore, we show the integrability of this class by transforming it into a Bernoulli differential equation. Then we determine the sufficient conditions for which a non-algebraic limit cycle enclosing the origin occurs. This limit cycle is given analytically.

As a main result, we prove:

Theorem 1. *Let $\mathcal{X} = (P, Q)$ the planar vector field given by (1). Then the following statements hold*

(i) *$\mathcal{X}$ has no singular points at infinity.*

(ii) *The origin is a degenerate linearly zero finite singular point topologically equivalent to a node.*

(iii) *$\mathcal{X}$ is integrable and its first integral is*

$$I(x,y) = \frac{(x^2+y^2)}{\exp\left(\int\limits_0^{\arctan\frac{y}{x}} \frac{f(u)}{g(u)}du\right)} - \int\limits_0^{\arctan\frac{y}{x}} \frac{\frac{4c}{g(u)}}{\exp\left(\int\limits_0^{u} \frac{f(s)}{g(s)}ds\right)}du,$$

(V) *For $a < 0, c > 0$, $\mathcal{X}$ has one and only one non-algebraic limit cycle, given in polar coordinates (r, θ) by*

$$r(\theta, r_*) = \exp\left(\frac{1}{2}\int\limits_0^{\theta} \frac{f(u)}{g(u)}du\right)\left(r_*^2 + \int\limits_0^{\theta} \frac{\frac{4c}{g(u)}}{\exp\left(\int\limits_0^{u} \frac{f(s)}{g(s)}ds\right)}du\right)^{\frac{1}{2}},$$

with

$$r_* = \left(\frac{\int\limits_0^{2\pi} \frac{\frac{4c}{g(u)}}{\exp(\int\limits_0^u \frac{f(s)}{g(s)} ds)} du}{\exp\left(-\int\limits_0^{2\pi} \frac{f(u)}{g(u)} du \right) - 1} \right)^{\frac{1}{2}},$$

where

$$f(\theta) = 3a + 4\sin 4\theta + a\cos 4\theta + 4b\sin 2\theta,$$
$$g(\theta) = 3 + \cos 4\theta,$$

are continuous real functions.

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