Averaging methods for piecewise differential systems formed by a linear focus or center and a cubic weak focus or center

M. Barkat[1], **R. Benterki**[2]

[1,2]Department of Mathematics,, University Mohamed El Bachir El Ibrahimi, Bordj Bou Arreridj 34000, El Anasser, Algeria

meriem.barkat@univ-bba.dz, r.benterki@univ-bba.dz

Finding the number of limit cycles, as described by Poincare [1], is one of the main problems in the qualitative theory of real planar differential systems. In general, studying limit cycles is a very challenging problem that is frequently difficult to solve.

In this paper, we are interested in finding an upper bound for the maximum number of limit cycles bifurcating from the periodic orbits of a given discontinuous piecewise differential system when it is perturbed inside a class of polynomial differential systems of the same degree, by using the averaging method up to third order. We prove that the discontinuous piecewise differential systems formed by a linear focus or center and a cubic weak focus or center separated by one straight line can have at most 7 limits.

We deal with polynomial differential systems in $\mathbb{R}^2$ of the form

$$\frac{dx}{dt} = P(x,y), \qquad \frac{dy}{dt} = Q(x,y), \tag{1}$$

where the degree of the systems is the maximum degree of P and Q. The second part of the 16th Hilbert problem proposes to find an upper bound for the maximum number of limit cycles and relative configurations for the differential system (1). We recall that a limit cycle of the differential system (1) is an isolated periodic orbit in the set of all periodic orbits of the system.

The objective of this paper is to study the limit cycles that can bifurcate from the discontinuous piecewise differential systems separated by the straight line $y = 0$ and formed by a linear differential system having a center or focus of the form

$$\dot{x} = \alpha x + \beta y + \gamma, \qquad \dot{y} = -\beta x + \alpha y + \delta. \tag{2}$$

defined in the half-plane $y \geq 0$, where α, β, γ, and $\delta \in \mathbb{R}$, and by an arbitrary cubic weak focus or center located at the origin

$$\begin{aligned} \dot{x} = & \ -y - ax^2 - cxy - zy^2 - kx^3 - mx^2y - pxy^2 - hy^3, \\ \dot{y} = & \ x + by^2 + dxy + gx^2 + ly^3 + nxy^2 + qx^2y + wx^3. \end{aligned} \tag{3}$$

defined in the half-plane $y \leq 0$, where all the parameters of the system are real.

Theorem 1. *For $\varepsilon \neq 0$ sufficiently small the maximum number of limit cycles of the discontinuous piecewise differential systems formed by linear differential focus or center* (2) *and the cubic weak focus or center* (3) *is at most seven limit cycles bifurcating from the periodic orbits of these systems.*

We consider the following discontinuous differential system

$$\dot{r}(\theta) = \begin{cases} F^+(\theta, r, \varepsilon) \ \ if \ \ 0 \leq \theta \leq \pi, \\ F^-(\theta, r, \varepsilon) \ \ if \ \ \pi \leq \theta \leq 2\pi. \end{cases} \tag{4}$$

where $F^{\pm}(\theta, r, \varepsilon) = \sum_{i=0}^{3} \varepsilon^i F_i^{\pm}(\theta, r) + \varepsilon^4 R^{\pm}(\theta, r, \varepsilon)$, $\theta \in \mathcal{S}^1$ and $r \in D$ where D is an open interval of $\mathbb{R}^+$.

A basic question in the study of discontinuous differential systems (4) is to understand which periodic orbits of the unperturbed system $\dot{r}(\theta) = F^{\pm}(\theta, r)$ persists for $|\varepsilon| \neq 0$ sufficiently small. We define a collection of functions $f_i : D \longrightarrow \mathbb{R}$ for $i = 1, 2, \ldots, k$, called averaged functions, such that their simple zeros provide the existence of isolated periodic solutions of the differential equation (4), these averaged functions are given by $f_i = \dfrac{y_i(2\pi, r)}{i!}$ where $y_i : \mathbb{R} \times D \to \mathbb{R}$, are defined by the following integrals

$$\begin{aligned}
y_1^{\pm}(s, r) = &\int_0^s F_1^{\pm}(t, r)dt,\\
y_2^{\pm}(s, r) = &\int_0^s \Big(2F_2^{\pm}(t, r) + 2\partial F_1^{\pm}(t, r) y_1^{\pm}(t, r)\Big)dt,\\
y_3^{\pm}(s, r) = &\int_0^t \Big(6F_3^{\pm}(t, r) + 6\partial F_2^{\pm}(t, r) y_1^{\pm}(t, z) + 3\partial^2 F_1^{\pm}(t, r) y_1(t, r)^2 + 3\partial F_1^{\pm}(t, r)\\
&y_2^{\pm}(t, r)\Big)dt.
\end{aligned}$$

Also, we have the functions

$$\begin{aligned}
f_1^{\pm}(r) = &\int_0^{\pm\pi} F_1^{\pm}(t, r)dt,\\
f_2^{\pm}(r) = &\int_0^{\pm\pi} \Big(F_2^{\pm}(t, r)dt + \partial F_1^{\pm}(t, r) y_1^{\pm}(t, r)\Big)dt,\\
f_3^{\pm}(r) = &\int_0^{\pm\pi} \Big(F_3^{\pm}(t, r)dt + \partial F_2^{\pm}(t, r) y_1^{\pm}(t, r) + \frac{1}{2}\partial^2 F_1^{\pm}(t, r) y_1^{\pm}(t, r)^2 + \frac{1}{2}\partial F_1^{\pm}(t, r)\\
&y_2^{\pm}(t, r)\Big)dt.
\end{aligned}$$

For more details see [2].

The averaged function of order k is the function $f_k(r) = f_k^{+}(r) + f_k^{-}(r)$. The simple positive real roots of the functions $f_{l+1}(r)$ which satisfy $f_l(r) = 0$ for $l \in \{1, 2\}$ but $f_{l+1}(r) \neq 0$, provide limit cycles of the piecewise differential system (4).

We need to state the Descartes Theorem in order to demonstrate our results regarding the number of zeros in a real polynomial.

Theorem 2 (Descartes theorem [3])**.** *Consider the real polynomial $r(x) = a_{i_1}x^{i_1} + a_{i_2}x^{i_2} + \ldots + a_{i_r}x^{i_r}$ with $0 = i_1 < i_2 < \ldots < i_r$ and $a_{i_j} \neq 0$ real constant for $j \in \{1, \ldots, r\}$. When $a_{i_j}a_{i_{j+1}} < 0$, we say that a_{i_j} and $a_{i_{j+1}}$ have a variation of the sign. If the number of variations of signs is m, then $r(x)$ has at most m positive real roots. Moreover, it is always possible to choose the coefficients of $r(x)$ in such a way that $r(x)$ has exactly $r-1$ positive real roots.*

1. Poincaré M. Memoire sur les coubes définies par une équation différentielle. — Sceaux: Editions Jacques Gabay, 1993.
2. Itikawa J., Llibre J., Novaes D. D. A new result on averaging theory for a class of discontinuous planar differential systems with applications. Rev. Mat. Iberoam, 2017, 33, 1247–1265.
3. Berezin I. I., Zhidkov N. Computing methods. Volume II. — Oxford: Pergamon Press, 1964.