Homoclinic solutions and Solitons in the discrete nonlinear Helmholtz-Schrodinger equation

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In this talk, we use the homoclinic orbit approach without using small perturbations to prove the existence of soliton solutions of the discrete nonlinear Schrodinger equations with Helmholtz operator by employing the properties of the symmetries of reversible planar maps.

In the past decade, the existence of discrete solitons of the DNLS equations has become a hot topic. Among the methods used are variational methods, center manifold reduction and Nehari manifold approach. Many of papers considered the DNLS equations with constant coefficients see [1,4,5]. Recently, the DNLS equations with periodic coefficients have appeared in physics literature for example [3], and such phenomenon can be found by numerical simulation. We look at the the existence of time periodic and spatially localized solutions in which a complex amplitude algebraic equation is obtained.

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The talk is organized as follows: First of all we give some preliminaries about the reversible planar map and the homocline (heterocline) points [2]. In addition we present the fondamental theorem for the existing of the the orbit homocline (heterocline) for a class of planar map in n dimension [3,5]. Next, we give the conditions to proove the existing the bright and dark dark soliton for local solutions in the of discret schrodiger equations with Helmholtz operator by employing the properties of the symmetries of reversible planar maps.

We consider the difference expression

$$L_n x_n = \tfrac{1}{w_n}(x_{n+1} + x_{n-1} + d_n x_n).$$

A class of classical reversible planar maps are derived from symmetric difference equations of the form:

$$x_{n+1} + x_{n-1} = g(x_n, w_n, d_n, h_n).$$

Let $f(z) = g(z) - dz$.

Theorem 1. *Assume that*
1- $f(z)$ is a $\mathcal{C}^1$ and odd function, and has only three real zeros, $-z_0, 0,$ and $z_0 (z_0 > 0)$ with $f(0) > 0$;
2- $sup_{z \geq z_1}((d-2)z + f(z) < 0$ for some $z' > z$. Then the planar map T has a homoclinic orbit.

Theorem 2. *Assume that*
1- $f(z)$ is a $\mathcal{C}^1$ and odd function, and has only three real zeros, $-z_0, 0,$ and $z_0 (z_0 > 0)$ with $f'(0) < -2d$,
2- $inf_{z \geq z'}(\{f(z) + 2dz\}) > 0$, for some $z' > z_0$.
Then the planar map T has a homoclinic orbit.

We investigate the DNLS equations with Helmholtz operator and general nonlinearities:

$$i\frac{\partial \psi_n}{\partial t} + h(\mid \psi_n \mid)\psi_n + \frac{1}{w_n}(\psi_{n+1} + \psi_{n-1} - d_n\psi_n) = 0 \tag{1}$$

where h is C^1 function.
Great attention has been paid to localized solutions of the form $\psi_n = x_n e^{-i\Omega t}$ where x_n are time independent. Such solutions are time periodic and spatially localized. We leads to a difference equation

$$-\Omega x_n + h(\mid x_n \mid)x_n + \frac{1}{w_n}(x_{n+1} + x_{n-1} - d_n x_n) = 0$$

$$f(z) = \omega(\Omega - h(\mid z \mid))z.$$

Theorem 3. *1-Assume that h is strictly increasing in $[0, +\infty[$. Then there exist an unstaggered (staggered) bright Solitons of the form $x_n e^{i\Omega t}$ with $h(0) < \Omega < h_\infty$ ($h(0) - 2d/w < \Omega < h_\infty - 2d/w$) for the system (1) with $w > 0$.*
2-Assume that h is strictly decreasing in $[0, +\infty[$. Then there exist a bright Solitons of the form $x_n e^{i\Omega t}$ with $h_\infty < \Omega < h(0)$ for the system (1) with $w < 0$.

Similarly the other cases can proved.

Theorem 4. *Assume that $h'(r) > 0$ (< 0) for $r \in [0, +\infty[$. Then there exist a dark Solitons of the form $x_n e^{i\Omega t}$ with $h(0) < \Omega < h_\infty$ ($h_\infty < \Omega < h(0)$) for the system (1) with $w < 0$ (> 0).*

We are interesting in the possibility of finding non-trivial homoclinic solutions for (1). This problem comes up when we look for the discrete solutions of the periodic equation DNLS:

$$h(\mid \psi_n \mid) = \frac{\sigma\chi_n \mid \psi_n \mid^2}{1 + c_n \mid \psi_n \mid^2}$$

DNLS with saturable nonlinearities can be used to describe the propagation of optical pulses in different doped fibers.

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