Nevanlinna theory in a punctured disc and applications to linear differential equations

S. Mazouz[1], **S. Hamouda**[2]
[1,2]Abdelhamid Ibn Badis University, Mostaganem, Algeria
[1]*said.mazouz.etu@univ-mosta.dz,* [2]*saada.hamouda@univ-mosta.dz*

In this work, we will investigate the growth and oscillation, near the singular point $z = 0$, of solutions to the differential equation

$$f'' + \left(A(z)\exp\left\{\frac{a}{z^n}\right\} + A_0(z)\right)f' + \left(B(z)\exp\left\{\frac{b}{z^n}\right\} + B_0(z)\right)f = H(z),$$

where $A(z), A_0(z), B(z), B_0(z), H(z)$ are analytic functions in

$$D(0,R) = \{z \in \mathbb{C} : 0 < |z| < R\}$$

and a, b are non-zero complex constants.

The idea to study the growth of solutions of the linear differential equations near a finite singular point by using the Nevanlinna theory has began by the paper [2]; then after some publications have followed, see [1,3]; (for the fundamental results, the definitions and the standard notations of the Nevanlinna theory see [5]). The principal tools used in these investigations is the estimates of the logarithmic derivative $\left|\frac{f^{(k)}(z)}{f(z)}\right|$ for a meromorphic function f in $\overline{\mathbb{C}} \setminus \{z_0\}$, $\left(\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}\right)$. A question was asked in [2,3] about if we can get similar estimates near z_0 of $\left|\frac{f^{(k)}(z)}{f(z)}\right|$ where f is a meromorphic function in a region of the form $D_{z_0}(0,R) = \{z \in \mathbb{C} : 0 < |z - z_0| < R\}$. This question was answered in [4]. In this talk, we will give some applications of these estimates on the growth and oscillation of solutions of certain class of linear differential equations with analytic coefficients in a punctured disc. In fact, we will prove the following results.

Theorem 1. *Let $A(z) \not\equiv 0, B(z) \not\equiv 0, F(z)$ be analytic functions in $D(0,R)$ such that $\max\{\sigma(A,0), \sigma(B,0), \sigma(F,0)\} < n, n \in \mathbb{N} \setminus \{0\}$; let a, b be complex constants such that $ab \neq 0$ and $a \neq b$. Then, every solution $f(z) \not\equiv 0$ of the differential equation*

$$f'' + A(z)\exp\left\{\frac{a}{z^n}\right\}f' + B(z)\exp\left\{\frac{b}{z^n}\right\}f = F(z), \tag{1}$$

satisfies $\sigma(f,0) = \infty$. Further, if $F(z) \not\equiv 0$, we have

$$\bar{\lambda}(f,0) = \lambda(f,0) = \sigma(f,0) = +\infty,\ \bar{\lambda}_2(f,0) = \lambda_2(f,0) = \sigma_2(f,0) \leq n.$$

Theorem 2. *Let $A(z) \not\equiv 0, A_0(z), B(z) \not\equiv 0, B_0(z), F(z)$ be analytic functions in $D(0,R)$ such that*

$$\max\{\sigma(A_0,0), \sigma(B_0,0), \sigma(A,0), \sigma(B,0), \sigma(F,0)\} < n,\ n \in \mathbb{N} \setminus \{0\};$$

let a, b be complex constants such that $ab \neq 0$ and $a = cb,\ c < 0$. Then, every solution $f(z) \not\equiv 0$ of the differential equation

$$f'' + \left(A(z)\exp\left\{\frac{a}{z^n}\right\} + A_0(z)\right)f' + \left(B(z)\exp\left\{\frac{b}{z^n}\right\} + B_0(z)\right)f = F(z), \tag{2}$$

satisfies $\sigma(f,0)=\infty$. *Further, if* $F(z)\not\equiv 0$, *we have*

$$\bar{\lambda}(f,0)=\lambda(f,0)=\sigma(f,0)=+\infty,\ \bar{\lambda}_2(f,0)=\lambda_2(f,0)=\sigma_2(f,0)\le n.$$

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