Wiman-Valiron theory of entire functions and linear fractional differential equations

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In this work, we investigate the order of growth of solutions to certain class of linear fractional differential with entire coefficients involving the Caputo fractional derivatives; for the definition and properties of different type of fractional derivatives see [3,5]. To this end, we use the generalized Wiman-Valiron theorem in the fractional calculus [1] and the Nevanlinna theory of meromorphic functions [4].

Definition 1. The order of growth of an entire function $f(z)$ is defined by

$$\sigma(f) = \limsup_{r\to+\infty} \frac{\log^+ m(r,f)}{\log r},$$

where

$$m(r,f) = \frac{1}{2\pi}\int_0^{2\pi} \ln^+ \left| f\left(re^{i\varphi}\right)\right| d\varphi;$$

and we have

$$\sigma(f) = \limsup_{r\to+\infty} \frac{\log^+ \log^+ M(r,f)}{\log r},$$

where $M(r,f) = \max\{|f(z)| : |z| = r\}$.

The authors have investigated in [2] the growth of solutions of certain class of linear fractional differential equations with polynomial coefficients. This talk is devoted to the study of linear fractional differential equations with entire coefficients. The main results of this work are as follows.

Theorem 1. *Let $A_0(z), A_1(z), ..., A_{n-1}(z)$ be entire functions and $\rho > 0,\ \delta > 0$, be constants such that $\max\{\sigma(A_j) : j = 1, ..., n-1\} < \rho$ and $A_0(0) = 0$; let $0 < q_1 < q_2 < ... < q_n$. Suppose that for any $\theta \in [0, 2\pi)$*

$$\left|A_0\left(re^{i\theta}\right)\right| \geq \exp\{\delta r^\rho\} \tag{1}$$

as $r \to +\infty$. Then, every solution $f(z) \not\equiv 0$ of the linear fractional differential equation

$$\frac{r^{q_n}}{z^{[q_n]}}\mathcal{D}^{q_n} f(z) + A_{n-1}(z)\frac{r^{q_{n-1}}}{z^{[q_{n-1}]}}\mathcal{D}^{q_{n-1}} f(z) + ... + A_1(z)\frac{r^{q_1}}{z^{[q_1]}}\mathcal{D}^{q_1} f(z) \tag{2}$$

$$+zA_0(z) f(z) = 0.$$

is entire function of infinite order and further if $\sigma(A_0) < \infty$ then $\sigma_2(f) \leq \sigma(A_0)$.

Theorem 2. *Let $A_1(z), B(z), A_0(z) \not\equiv 0$, be entire functions and let $\rho > 0$, $\delta > 0$, $0 < \alpha < 1$ be constants such that $\max\{\sigma(A_0), \sigma(B)\} < \rho$. Suppose that there exists a set $E \subset [0, 2\pi)$ of linear measure zero such that for any $\theta \in [0, 2\pi) \setminus E$*

$$\left|A_1\left(re^{i\theta}\right)\right| \geq \exp\{\delta r^{\rho}\} \tag{3}$$

as $r \to +\infty$. Then every solution $f(z) \not\equiv 0$ of the differential equation

$$e^{-2i\theta}\mathcal{D}^2 f(z) + A_1(z) e^{-i\theta}\mathcal{D}^1 f(z) + B(z) r^{\alpha}\mathcal{D}^{\alpha} f(z) + A_0(z) f(z) = 0 \tag{4}$$

is an entire function of infinite order and further if $\sigma(A_1) < \infty$ then $\sigma_2(f) \leq \sigma(A_1)$.

Theorem 3. *Let $A_1(z), B(z), A_0(z) \not\equiv 0, F(z)$ be entire functions and let $\rho > 0$, $\delta > 0$, $0 < \alpha < 1$ be constants such that $\max\{\sigma(A_0), \sigma(B), \sigma(F)\} < \rho$. Suppose that there exists a set $E \subset [0, 2\pi)$ of linear measure zero such that for any $\theta \in [0, 2\pi) \setminus E$*

$$\left|A_1\left(re^{i\theta}\right)\right| \geq \exp\{\delta r^{\rho}\} \tag{5}$$

as $r \to +\infty$. Then every solution $f(z) \not\equiv 0$ of the differential equation

$$e^{-2i\theta}\mathcal{D}^2 f(z) + A_1(z) e^{-i\theta}\mathcal{D}^1 f(z) + B(z) \frac{r^{\alpha}}{z^{[\alpha]}}\mathcal{D}^{\alpha} f(z) + A_0(z) f(z) = F(z) \tag{6}$$

is an entire function of infinite order.

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