Global asymptotic stability of an epidemic reaction-diffusion model

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The aim of this work is to study the global asymptotic stability of a reaction-diffusion SI (susceptible-infectious) epidemic model with a nonlinear incidence rate describing the transmission of a communicable disease between individuals.We prove that the proposed model has two steady states under one condition. By an appropriately constructed Lyapunov functional, we establish the global asymptotic stability of the non-negative constant steady states subject to the basic reproduction number being greater than unity and of the disease-free equilibrium subject to the basic reproduction number being smaller than or equal to unity in ODE case. By applying an appropriately constructed Lyapunov functional, we identify the condition of the global stability in the PDE case. Finally, we present some numerical examples illustrating and confirming the analytical results obtained throughout the paper.

In this manuscript, we consider the reaction–diffusion epidemic phenomenon proposed in [1], which is an extended version of the SIS epidemic model with the nonlinear incidence $u\varphi(v)$. The system is described as

$$\begin{cases} \dfrac{\partial u}{\partial t} - d_1\Delta u = \Lambda - \mu u - \lambda u\varphi(v) := F(u,v) & \text{in } (0,\infty)\times\Omega, \\ \dfrac{\partial v}{\partial t} - d_2\Delta v = -\sigma v + \lambda u\varphi(v) := G(u,v) & \text{in } (0,\infty)\times\Omega. \end{cases} \tag{1}$$

Throughout this paper, the notation Δ denotes the Laplacian operators on Ω, where Ω is an open bounded subset of $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. The constant parameters $d_1, d_2 \geq 0$ are the diffusion coefficients. We assume the initial conditions

$$u_0(x) = u(x,0),\ v_0(x) = v(x,0) \qquad \text{in } \Omega, \tag{2}$$

where $u_0, v_0 \in C(\overline{\Omega})$, and impose homogeneous Neumann boundary conditions

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \qquad \text{on } (0,\infty)\times\partial\Omega, \tag{3}$$

with ν being the unit outer normal to $\partial\Omega$. We will also assume that the initial conditions $u_0(x), v_0(x) \in \mathbb{R}_{\geq 0}$. For the purpose of this study, we will assume that $\mu, \sigma, \lambda > 0$. The incidence function $\varphi(v)$ introduces a nonlinear relation between the two classes of individuals. We assume φ to be a continuously differentiable function on $\mathbb{R}^+$ satisfying

$$\varphi(0) = 0, \tag{4}$$

and

$$0 < v\varphi'(v) \leq \varphi(v) \quad \text{for all } v > 0. \tag{5}$$

First, however, we state a lemma that was developed in [2] and which will become useful later on.

Lemma 1. *Condition (5) implies*

$$0 < \frac{\varphi(v)}{v} \leq \varphi'(0) \text{ for all } v > 0.$$

Theorem 1. *System (1)–(3) has a single disease–free equilibrium* $E_0 = \left(\frac{\Lambda}{\mu}, 0\right)$. *If the basic reproduction number* $R_0 = \frac{\Lambda\lambda}{\mu\sigma}\varphi'(0) > 1$, *then the system admits two distinct equilibria:* E_0 *and the positive endemic equilibrium* $E^* = (u^*, v^*) \in \mathbb{R}^2_{>0}$.

Theorem 2. *If* $R_0 < 1$, E_0 *is a globally asymptotically stable disease–free steady state for system (1)–(3) under the assumption*

$$\varphi'(0) \leq \frac{\mu + \sigma}{\lambda\left(\theta\frac{\Lambda}{\mu} + \frac{\Lambda}{\sigma}\right)},$$

with

$$\theta \geq \frac{(d_1 + d_2)^2}{4d_1d_2}.$$

Theorem 3. *If* $R_0 > 1$, E^* *is a globally asymptotically stable endemic steady–state for system (1)–(3).*

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