Pseudo-discrete symmetries

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Some discrete point symmetries of a system of differential equations can easily be guessed but it is usually a challenging problem to list all its independent discrete symmetries, which may need the explicit construction of the entire point symmetry (pseudo)group of this system. Even for so classical, most fundamental and well-studied models of mathematical physics as $(1+1)$-dimensional linear heat equation

$$u_t = u_{xx}, \tag{1}$$

their discrete point symmetries have not been completely understood. Thus, the involution $\mathscr{J}$ alternating the sign of x and the transformation $\mathscr{K}'$ inverting t,

$$\mathscr{J} := (t,x,u) \mapsto (t,-x,u), \qquad \mathscr{K}'\colon\quad \tilde t = -\frac{1}{t},\ \ \tilde x = \frac{x}{t},\ \ \tilde u = \sqrt{|t|}\,\mathrm{e}^{\frac{x^2}{4t}}u$$

were for a long time considered as discrete symmetries of (1). However, in [1] we show that they both belong not only to the identity component of the point symmetry pseudogroup G of (1) but, moreover, to a one-parameter subgroup of G. In contrast to G, the maximal Lie invariance algebra $\mathfrak g$ of (1) is well known, $\mathfrak g = \mathfrak g^{\rm ess} \Subset \mathfrak g^{\rm lin}$, where $\mathfrak g^{\rm ess} = \langle \mathcal P^t, \mathcal D, \mathcal K, \mathcal G^x, \mathcal P^x, \mathcal I\rangle$, $\mathfrak g^{\rm lin} = \{\mathcal Z(h)\}$ is an infinite-dimensional abelian ideal of $\mathfrak g$, and h runs through the solution set of (1).

Theorem 1. ([1,2]) *The point symmetry pseudogroup G of the (1+1)-dimensional linear heat equation (1) consists of the point transformations of the form*

$$\tilde t = \frac{\alpha t+\beta}{\gamma t+\delta},\quad \tilde x = \frac{x+\lambda_1 t+\lambda_0}{\gamma t+\delta},\quad \tilde u = \sigma\sqrt{|\gamma t+\delta|}\,\exp\!\left(\frac{\gamma(x+\lambda_1 t+\lambda_0)^2}{4(\gamma t+\delta)} - \frac{\lambda_1}{2}x - \frac{\lambda_1^2}{4}t\right)\big(u+h(t,x)\big),$$

where α, β, γ, δ, λ_1, λ_0 and σ are arbitrary constants with $\alpha\delta-\beta\gamma=1$ and $\sigma\neq 0$, and h is an arbitrary solution of (1).

We modify the standard operation in G by continuously extending the domains of transformation compositions and then properly analyze the structure of G [1]. The pseudogroup G splits over $G^{\rm lin} := \{\mathscr Z(h)\colon \tilde t = t,\ \tilde x = x,\ \tilde u = u + h(t,x)\} \lhd G$, $G = G^{\rm ess} \ltimes G^{\rm lin}$, where the subgroup $G^{\rm ess}$ of G is singled out by the constraint $h=0$ and, therefore, is a six-dimensional Lie group. Moreover, $G^{\rm ess} = F \ltimes R$, where $R \lhd G^{\rm ess}$ and $F < G^{\rm ess}$ are singled out by the constraints $\alpha=\delta=1$, $\beta=\gamma=0$ and $\lambda_1=\lambda_0=0$, $\sigma=1$, respectively. $R \simeq \mathrm H(1,\mathbb R)\times\mathbb Z_2$ and $F\simeq \mathrm{SL}(2,\mathbb R)$, where $\mathrm{SL}(2,\mathbb R)$ and $\mathrm H(1,\mathbb R)$ denote the real degree-two special linear group and the real rank-one Heisenberg group, respectively.

In fact, the transformation $\mathscr J$ belongs to the subgroup F of $G^{\rm ess}$ and under the isomorphism $F\to \mathrm{SL}(2,\mathbb R)$ the image of $\mathscr J$ is the matrix $-E$, where $E := \mathrm{diag}(1,1)$. The matrix $-E$ has very peculiar properties.

Recall that the matrix $A\in\mathrm{SL}(2,\mathbb R)$ with $A\ne\pm E$ is called *elliptic* if $|\operatorname{tr}A|<2$, *parabolic* if $|\operatorname{tr}A|=2$ and *hyperbolic* if $|\operatorname{tr}A|>2$. It is well known that a matrix $A\in\mathrm{SL}(2,\mathbb R)$ belongs to $\exp\big(\mathrm{sl}(2,\mathbb R)\big)$ if and only if either $\operatorname{tr}A>-2$ or $A=-E$. Hence, the set $\exp\big(\mathrm{sl}(2,\mathbb R)\big)$ consists of the elliptic elements, the hyperbolic and parabolic elements with positive traces, E and $-E$. Since the multiplication by $-E$ switches the signs of matrix traces, it maps the hyperbolic and parabolic parts of $\exp\big(\mathrm{sl}(2,\mathbb R)\big)$ onto the complement of $\exp\big(\mathrm{sl}(2,\mathbb R)\big)$ in $\mathrm{SL}(2,\mathbb R)$, and vice versa. Therefore, the action by $-E$ changes the relation "belongs to (no) one-parameter subgroup" to its negation for hyperbolic and parabolic elements of $\mathrm{SL}(2,\mathbb R)$, and in this sense the role of $-E$ on the level of one-parameter subgroups is analogous to the role of a discrete element on the level of connected components. This is why we call $-E$ a *pseudo-discrete* element of $\mathrm{SL}(2,\mathbb R)$.

Definition 1. Given a Lie group G with Lie algebra $\mathfrak g$ and $\exp(\mathfrak g)\ne G_{\rm id}$, we call an element $g\in G$ *pseudo-discrete* if $g\big(G_{\rm id}\setminus\exp(\mathfrak g)\big)\subseteq\exp(\mathfrak g)$.

Proposition 1. *The transformation $\mathscr J$ is a pseudo-discrete element of $G^{\rm ess}$.*

Given a system of differential equations, it is natural to call pseudo-discrete elements of its point symmetry (pseudo)group *pseudo-discrete point symmetries* of this system. So far the question if the transformation $\mathscr J$ can be called a pseudo-discrete point symmetry of the equation (1) is an open problem. However, an example of a genuine pseudo-discrete point symmetry is given by the Burgers equation

$$v_t+vv_x=v_{xx},\tag{2}$$

whose maximal Lie invariance algebra is $\mathfrak g^{\rm B}=\langle\breve{\mathcal P}^t,\breve{\mathcal D},\breve{\mathcal K},\breve{\mathcal P}^x,\breve{\mathcal G}^x\rangle$.

Theorem 2. ([1,3]) *The point symmetry group $G^{\rm B}$ of the Burgers equation (2) is constituted by the point transformations of the form*

$$\tilde t=\frac{\alpha t+\beta}{\gamma t+\delta},\quad \tilde x=\frac{x+\lambda_1t+\lambda_0}{\gamma t+\delta},\quad \tilde v=(\gamma t+\delta)v-\gamma x+\lambda_1\delta-\lambda_0\gamma,$$

where α, β, γ, δ, λ_1 and λ_0 are arbitrary constants with $\alpha\delta-\beta\gamma=1$.

Proposition 2. *$\breve{\mathscr J}\big(G^{\rm B}\setminus\exp(\mathfrak g^{\rm B})\big)\subseteq\exp(\mathfrak g^{\rm B})$ for $\breve{\mathscr J}:=(t,x,v)\mapsto(t,-x,v)\in G^{\rm B}$, i.e., this transformation is a pseudo-discrete element of $G^{\rm B}$ or, equivalently, is a pseudo-discrete point symmetry of the equation (2).*

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