Well-posedness of solution to a coupled system wave/Wentzell with nonlinear dampings and delays

L. Ihaddadene[1]

[1] University of Science and Technology Houari Boumediene, Algiers, Algeria
lilaihaddadene5@gmail.com

In this work, we study the existence and the uniqueness of the solution of a wave equation with dynamic Wentzell type boundary conditions on a part of the boundary Γ_1 of the domain Ω, with nonlinear delays in nonlinear dampings in Ω and on Γ_1, given by:

$$\begin{cases} u_{tt} - \Delta u + \mu_1 g_1(u_t) + \mu_2 g_1(u_t(t-\tau)) = 0 \text{ in } \Omega \times (0,\infty), \\ v_{tt} + \partial_\nu u - \Delta_T v + \mu_1' g_2(v_t) + \mu_2' g_2(v_t(t-\tau)) = 0 \text{ on } \Gamma_1 \times (0,\infty) \\ u = v \text{ on } \Gamma \times (0,\infty) \\ u = 0 \text{ on } \Gamma_0 \times (0,\infty) \\ (u(0), v(0)) = (u_0, v_0) \text{ in } \Omega \times \Gamma, (u_t(0), v_t(0)) = (u_1, v_1) \text{ in } \Omega \times \Gamma \\ u_t(x, t-\tau) = f_{0_1}(x, t-\tau) \text{ in } \Omega \times (0,\tau), \\ v_t(x, t-\tau) = f_{0_2}(x, t-\tau) \text{ on } \Gamma_1 \times (0,\tau), \end{cases} \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $(n \geq 2)$, with smooth boundary $\Gamma = \partial\Omega$, divided into two closed and disjoint subsets Γ_0 and Γ_1, such that $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$ and $\Gamma_0 \cup \Gamma_1 = \Gamma$. We denote by ∇_T the tangential gradient on Γ, by Δ_T the tangential Laplacian on Γ and by ∂_ν the normal derivative where ν represents the unit outward normal to Γ. μ_1, μ_2, μ_1' and μ_2' are positive real numbers, the two functions $g_1(u_t(t-\tau))$ and $g_2(v_t(t-\tau))$ describe the delays on the nonlinear frictional dissipations $g_1(u_t)$ and $g_2(v_t)$, on Ω and Γ_1, respectively, $\tau > 0$ is a time delay and u_0, v_0, u_1, v_1, f_{0_1} and f_{0_2} are the initial data in some suitable (Sobolev) function spaces.
We introduce the following set

$$H^1_{\Gamma_0}(\Omega) = \left\{ u \in H^1(\Omega) /\ \ u|_{\Gamma_0} = 0 \right\},$$

which is endowed with the Hilbert structure induced by $H^1(\Omega)$.

Then, we consider the canonical norms of $H^1_{\Gamma_0}(\Omega)$ and $H^1(\Gamma_1)$

$$\|u\|^2_{H^1_{\Gamma_0}(\Omega)} = \|\nabla u\|^2_{L^2(\Omega)}, \qquad \|v\|^2_{H^1(\Gamma_1)} = \|\nabla_T v\|^2_{L^2(\Gamma_1)}.$$

Now, we state the following existence and uniqueness result:

Theorem 1. *Let* $(u_0, u_1, v_0, v_1) \in [H^2(\Omega) \cap H^1_{\Gamma_0}(\Omega)) \times H^1_{\Gamma_0}(\Omega)] \times [H^2(\Gamma_1) \times H^1(\Gamma_1)]$, $f_{0_1} \in H^1_{\Gamma_0}(\Omega; H^1(0,1))$ *and* $f_{0_2} \in H^1(\Gamma_1; H^1(0,1))$ *satisfying the following compatibility condition:*

$$\begin{cases} \partial_\nu u_0 - \Delta_T v_0 + \mu_1' g_2(v_1) = 0 \quad \text{on} \quad \Gamma_1, \\ f_{0_1}(\cdot, 0) = u_t \quad \text{in} \quad \Omega, \\ f_{0_2}(\cdot, 0) = v_t \quad \text{on} \quad \Gamma_1. \end{cases}$$

Then, problem (1) possesses a unique global weak solution verifying for $T > 0$:

$$(u, u_t, u_{tt}) \in L^\infty(0, T; \left[H^1_{\Gamma_0}(\Omega)\right]^2 \times L^2(\Omega)),$$

$$(v, v_t, v_{tt}) \in L^\infty(0, T; \left[H^1(\Gamma_1)\right]^2 \times L^2(\Gamma_1)).$$

We shall give a proof of above theorem, by using the Faedo-Galerkin's approximation (see [1, 2]).

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2. Ihaddadene L. Existence and Uniqueness of a Solution of a Wentzell's Problem with Nonlinear Delays. Comput. Sci. Math. Forum, 2023, 1, 0.