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Existence and regularity of solutions of nonlinear anisotropic elliptic problem with Hardy potential

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In this paper, we are interested in the existence and regularity of a solution for some anisotropic elliptic equations with Hardy potential and $L^m(\Omega)$ datum in appropriate anisotropic Sobolev spaces. The aim of this work is to get natural conditions to show the existence and regularity results for the solutions, that is related to a anisotropic Hardy inequality.

Let Ω be a bounded open set in $\mathbb{R}^N (N > 2)$ with smooth boundary $\partial\Omega$ and $\overrightarrow{p} = (p_1, ..., p_N)$ are restricted as follows

$$2 \leq p^- = \min_{1 \leq i \leq N}\{p_i\} < p_i < p^+ = \max_{1 \leq i \leq N}\{p_i\}, \quad 2 \leq \overline{p} < N, \quad \frac{1}{\overline{p}} = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{p_i} \tag{1}$$

The anisotropic Laplace operator $\Delta_{\overrightarrow{p}} u$ is defined by

$$\Delta_{\overrightarrow{p}} u = \sum_{i=1}^{N} D_i\left(|D_i u|^{p_i-2}D_i u\right), \quad D_i u = \frac{\partial u}{\partial x_i} \quad \forall i = \overline{1, N}.$$

This paper deals with the study of existence and regularity of solutions for a class of nonlinear anisotropic elliptic problems

$$\begin{cases} -\Delta_{\overrightarrow{p}} u = \frac{|u|^{p^- -1}}{|x|^{p^-}} + f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where f belongs to $L^m(\Omega)$ with $m \geq 1$ and $\mu > 0$, such that

$$0 \leq \mu < \frac{(\lambda-1)N^{\frac{p^-+2}{2}}}{M^{-1}_{p^+,p^-}}\left(\frac{p^-}{p^-+\lambda-2}\right)^{p^-}, \quad \lambda = \frac{N(m-1)(\overline{p}-1)+N-m\overline{p}}{N-m\overline{p}}. \tag{3}$$

Our main motive in this article is to investigate the results of [1] the framework of the operator non-homogeneous $\Delta_{\overrightarrow{p}} u$. To reach this goal, we will face the following difficulties. First, let us note that (2) can be singular at the origin on the right-hand side, the so-called Hardy potential. On the other hand, there is difficulty in applying anisotropic Hardy inequality, which plays a major role in showing the desired results. To overcome these difficulty we approximate the term $\frac{|u|^{p^- -1}}{|x|^{p^-}}$ by $\frac{|T_n(u_n)|^{p^- -1}}{\left(|x|+\frac{1}{n}\right)^{p^-}}$. Then, we prove the a priori estimates of the approximate solution sequence, by using the anisotropic Hardy inequality (see Theorem 1 and Corollary 1).

The problem (2) is related to the following Anisotropic Hardy type inequality (see [2]).

Theorem 1 ([2]). *Let $v \in C_0^1(B)$, $1 < p_i < N$, $i = \overline{1, N}$, $B = \{x \in \mathbb{R}^N; \text{such that } x_i \neq 0, \ \forall\, i = \overline{1, N}\}$. Then we have*

$$\sum_{i=1}^{N}\int_B |D_i v|^{p_i}dx \geq \sum_{i=1}^{N}\left(\frac{p_i-1}{p_i}\right)^{p_i}\int_B \frac{|v|^{p_i}}{|x_i|^{p_i}}dx \geq M_{p^-,p^+}\sum_{i=1}^{N}\int_B \frac{|v|^{p_i}}{|x_i|^{p_i}}dx,$$

where $M_{p^-,p^+} = \left(\frac{p^- -1}{p^+}\right)^{p^-}$.

Corollary 1. *Let* $v \in C_0^1(\Omega), 1 < p_i < N, i = \overline{1, N}$, *then we have*

$$\int_\Omega \frac{|v|^{p^-}}{\left(|x| + \frac{1}{n}\right)^{p^-}} dx \leq M_{p^-,p^+}^{-1} N^{-\frac{p^- + 2}{2}} \sum_{i=1}^N \int_\Omega |D_i v|^{p_i} dx + M_{p^-,p^+}^{-1} N^{-\frac{p^-}{2}} |\Omega|.$$

The first result deals with a given f which yields unbounded solutions in energy space $W_0^{1,(p_i)}(\Omega)$.

Theorem 2. *Assume that* (1), (3) *hold true. Let* $f \in L^m(\Omega)$, *such that*

$$\frac{N\overline{p}}{N(\overline{p} - 1) + \overline{p}} \leq m < \frac{N}{\overline{p}},$$

Then, there exists a weak solution $u \in L^s(\Omega) \cap W_0^{1,(p_i)}(\Omega)$ *to problem* (2), *where*

$$s = \frac{Nm(\overline{p} - 1)}{N - \overline{p}m}.$$

The next result deals with the case when the summability of f gives the existence of solution $u \in W_0^{1,(\eta_i)}$, with $1 < \eta_i < p_i$ for every $i = \overline{1, N}$.

Theorem 3. *Assume that* (1), (3) *hold true. Let* $f \in L^m(\Omega)$, *such that*

$$1 < m < \frac{N\overline{p}}{N(\overline{p} - 1) + \overline{p}},$$

and for all $i = \overline{1, N}$

$$\frac{\overline{p}(N - m)}{mN(\overline{p} - 1)} < p_i < \frac{\overline{p}(N - m)}{\overline{p}(N - m) - mN(\overline{p} - 1)}.$$

Then, there exists a weak solution $u \in L^{\overline{\eta}^*}(\Omega) \cap W_0^{1,(\eta_i)}(\Omega)$ *to problem* (2), *where*

$$\eta_i = \frac{Nm(\overline{p} - 1)}{\overline{p}(N - m)} p_i \quad \forall i = \overline{1, N}.$$

1. Boccardo L., Orsina L., Peral I. A remark on existence and optimal summability of solutions of elliptic problems involving Hardy potential. Discrete and Continuous Dynamical Systems, 2006, 16, No. 3, 513–523.
2. Tingfu F., Xuewei C. Anisotropic Picone identities and anisotropic Hardy inequalities. Journal of Inequalities and Applications, 2017, 2017, No. 13, 1–9.