Decay of solutions of a coupled system in a star-shaped domain with linear boundary and internal feedbacks

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We consider a parabolic/hyperbolic coupled system of two partial differential equations (PDEs), which features a suitable boundary and internal dissipation term. The system consists of the elasticity equation and the heat equation in a bounded domain Ω. We proved the exponential decay result with an estimation of the decay rates. Our result is established using the multiplier method [1].

let Ω be a bounded domain of $\mathbb{R}^3$ with a boundary $\Gamma = \partial\Omega$ of class C^3 such that $\Gamma = \Gamma_0 \cup \Gamma_1$. The model is given by:

$$\begin{cases} u'' - \mathbf{div}\sigma(u) + \xi\nabla\theta + a(x)u' = 0, & \text{in } Q = \Omega\times]0,\infty[, \\ \theta' - \Delta\theta + \beta\mathbf{div}u' = 0, & \text{in Q}, \\ \theta = 0, & \text{in Q}, \end{cases} \tag{1}$$

Our notations in (1) are standard:$u' = \frac{\partial u}{\partial t}$, $u'' = \frac{\partial^2 u}{\partial t^2}$, $u(x,t) \in \mathbb{R}^3$ denote the displacement vector at $x = (x_1, x_2, x_3) \in \Omega$ and t is the time variable and $\theta = \theta(x,t)$ represent the temperature. $\sigma(u) = (\sigma_{ij}(u))_{i,j=1}^3$ is the stress tensor given by $\sigma(u) = 2\alpha\varepsilon(u) + \lambda\mathbf{div}(u)I_3$, where λ and α are the Lamé coefficients, I_3 is the identity matrix of $\mathbb{R}^3$ and $\varepsilon(u) = \frac{1}{2}(\nabla u + (\nabla u)^T) = [\varepsilon_{ij}(u)]_{i,j=1}^3$ is a 3×3 symmetric matrix. From now on, a summation convention with respect to repeated indexes will be use. Also, in system (1) $a(x) \in L^\infty$ is a nonnegative coefficient of the damping term such that $a(x) \geq a_0 \geq 0$, the coupling parameters ξ and β are supposed to be positive and ρ, η are positive constants.

We complement system (1) with initial conditions

$$u(.,0) = u_0,\ u'(.,0) = u_1,\ \theta(.,0) = \theta_0, \text{ in } \Omega, \tag{2}$$

and Wentzell boundary conditions (see [3])

$$\begin{cases} u''_T + \sigma_S(u) - c^2\overline{\mathbf{div}_T\sigma^0_T(u)} + l(x)u_T + bu'_T = 0, & \text{on } \Sigma, \\ u''_\nu + \sigma_\nu(u) + \sigma^0_T(u) : \partial_m\nu + l(x)u_\nu + bu'_\nu = 0, & \text{on } \Sigma, \end{cases} \tag{3}$$

where $a = a(x)$ and $b = b(x)$ be two nonnegative functions belongs to $C^1(\Gamma)$, $\sigma^0_T(u) : \partial_m\nu = tr(\sigma^0_T(u) \cdot \partial_m\nu)$, with

$$\sigma^0_T(u) = 2\alpha\varepsilon^0_T(u) + \frac{2\lambda\alpha}{\lambda + 2\alpha} tr(\varepsilon^0_T(u))I_2,$$

and "tr" means the trace of a matrix. As usual $\nu = \nu(x)$ denotes the unit normal vector at $x \in \Gamma$ pointing the exterior of Ω.

We have the following existence and uniqueness result (see [2]):

Theorem 1. *The problem* $(1-2-3)$ *is well posed in the space* $\mathcal{H}$. *In particular: for all* $(u_0, u_1, \theta_0, u_{1|\Gamma}) \in D(\mathcal{A})$, *the problem* (1) *has a unique (strong) solution which satisfies* $(u, u', \theta, u'_{|\Gamma}) \in W^{1,\infty}(\mathbb{R}_+, \mathcal{H}) \cap L^\infty(\mathbb{R}_+, D(\mathcal{A}))$. *Moreover, for all* $(u_0, u_1, \theta_0, u_{1|\Gamma}) \in \mathcal{H}$, *the problem* $(1-2-3)$ *has a unique (weak) solution satisfying* $(u, u', \theta, u'_{|\Gamma}) \in \mathcal{C}(\mathbb{R}_+, \mathcal{H})$.

The main result of this work is the following theorem

Theorem 2. *Let (u,θ) be the weak solution of the coupled system $(1-2-3)$, then there exist two positive constants M and ω such that*

$$\mathcal{E}(t) \le \mathcal{E}(0)e^{1-\omega t}.$$

We begin by proving the following Lemma.

Lemma 1. *Suppose $q=(q_1,q_2,q_3)$ is a smooth vector field on Ω of class $W^{1,\infty}(\overline{\Omega})^3$. Then, for every strong solution (u,θ) of (1) we have the following identity:*

$$\begin{aligned}
0 = & -\left[\int_\Omega \underbrace{2\big(q:\nabla u\big)}_{M(u)} \cdot u' dx\right]_0^T + \int_{Q_T} \mathbf{div} q\big(\sigma(u):\varepsilon(u) - |u'|^2\big) dxdt \\
& - 2\int_{Q_T} \sigma(u)\cdot(\nabla q\cdot\nabla u)\, dxdt - \xi\int_{Q_T}\nabla\theta\cdot M(u)\, dxdt - \int_{Q_T} l(x)u'\cdot M(u)\, dxdt \\
& + \int_{\Sigma_T} \mathbf{div}_T q_T(\sigma_T^0(u):\varepsilon_T^0(u))d\Gamma dt + \int_{\Sigma_T} q\cdot\nu\Big(|u'|^2 + \alpha|\partial_\nu u_T|^2 + (2\alpha+\lambda)|\partial_\nu u_\nu|^2\Big)d\Gamma dt \\
& - \int_{\Sigma_T} q\cdot\nu[2\alpha\varepsilon_T^0(u) + \lambda tr(\varepsilon_T^0(u)i_2)] : \varepsilon_T^0(u) d\Gamma dt - \int_{\Sigma_T} q\cdot\nu(\alpha|\overline{\partial_T u_\nu} - (\partial_T\nu)u_T|^2)d\Gamma dt \\
& - 2\int_{\Sigma_T} \sigma_T^0(u):(\pi\partial_T u_T\pi + u_\nu\partial_T\nu)(\pi\partial_T q_T\pi)d\Gamma dt - 2\int_{\Sigma_T}\partial_T u_\nu(\sigma_T^0(u)(\partial_T\nu)q_T)d\Gamma dt \\
& + 2\int_{\Sigma_T}(\sigma_T^0(u):\partial_T\nu)\overline{u_T}(\partial_T\nu)q_T d\Gamma dt - 2\int_{\Sigma_T}(\mathcal{R}:\sigma_T^0(u)):(u_T\otimes q_T)d\Gamma dt \\
& - 2\int_{\Sigma_T}(au_T + bu')(\pi\partial_T u_T\pi + u_\nu\partial_T\nu)q_T d\Gamma dt - 2\int_{\Sigma_T}(au_\nu + bu')(\partial_T u_\nu - \overline{u_T}\partial_T\nu)q_T d\Gamma dt,
\end{aligned}$$

where $Q_T=[0,T]\times\Omega$, $\Sigma_T=[0,T]\times\Gamma$, $\mathcal{R}$ is the curvature tensor, once contravariant and three times covariant such that

$$\mathcal{R}^\mu_{\kappa\rho\varrho} = \Gamma^\mu_{\tau\kappa,\rho} - \Gamma^\mu_{\rho\kappa,\tau} + \Gamma^\mu_{\rho\varrho}\Gamma^\varrho_{\tau\kappa} - \Gamma^\mu_{\tau\varrho}\Gamma^\varrho_{\rho\kappa},$$

with $1\le\kappa,\rho,\varrho,\mu\le 2$ and $\Gamma^\theta_{\lambda\eta}$ are the Christoffel symbols (see for instance [?, ?]). For the sake of conciseness, we have used the notation $q:\nabla u = \Big(q\cdot\nabla u_1,\ q\cdot\nabla u_2,\ q\cdot\nabla u_3\Big)$ and $q_T:\nabla_T u = \Big(q_T\cdot\nabla_T u_1,\ q_T\cdot\nabla_T u_2,\ q_T\cdot\nabla_T u_3\Big)$, where q_T is the tangential component of q and $\mathbf{div}_T q_T$ is the tangential divergence of q_T.

1. Komornik V. Exact Controllability and Stabilization: The Multiplier Method. — RAM 36, Masson, Paris, 1994.
2. Pazy A. Semigroups of Linear Operators and Applications to Partial Differential Equations, Applied Mathematical Sciences, 44. — Springer-Verlag, 1983, 279 p.
3. Wentzell A. D. On boundary conditions for multi-dimensional diffusion processes. Theor. Probab. Appl, 1959, 4, 164–177.