On elliptic boundary-value problems in function spaces of low regularity

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Let Ω be a bounded Euclidean domain of dimension $n \geq 2$ and with boundary Γ of class C^∞. Choose integers $l \geq 1$, $\lambda \geq 0$, $m_1, \ldots, m_{l+\lambda} \leq 2l-1$, and $r_1, \ldots, r_\lambda$ if $\lambda \geq 1$. Consider an elliptic boundary-value problem of the form

$$Au = f \quad \text{in} \quad \Omega, \qquad B_j u + \sum\nolimits_{k=1}^{\lambda} C_{j,k} v_k = g_j \quad \text{on} \quad \Gamma, \quad j = 1, ..., l+\lambda.$$

Here, A is a linear partial differential operator (PDO) on $\overline{\Omega} = \Omega \cup \Gamma$ with $\operatorname{ord} A = 2l$; every B_j is a boundary PDO on Γ with $\operatorname{ord} B_j \leq m_j$, and each $C_{j,k}$ is a tangent PDO on Γ with $\operatorname{ord} C_{j,k} \leq m_j + r_k$. All coefficients of these PDOs are infinitely smooth complex-valued functions on $\overline{\Omega}$ and Γ, resp. Assume that $\max\{m_1, \ldots, m_{l+\lambda}\} \geq -r_k$ whenever $1 \leq k \leq \lambda$.

Let N denote the linear space of all solutions $(u, v_1, ..., v_\lambda) \in C^\infty(\overline{\Omega}) \times (C^\infty(\Gamma))^\lambda$ to problem () in which $f = 0$ on Ω and all $g_j = 0$ on Γ. Let N^+ stand for the linear space of all solutions $(w, h_1, ..., h_{l+\lambda}) \in C^\infty(\overline{\Omega}) \times (C^\infty(\Gamma))^{l+\lambda}$ to the formally adjoint problem in which all right-hand sides are zeros. The spaces N and N^+ are finite dimensional; put $\alpha := \dim N - \dim N^+$.

We study the solvability of the elliptic problem () in the complex Besov spaces $B^s_{p,q}$ and Triebel–Lizorkin spaces $F^s_{p,q}$ of order $s \leq 2l - 1 + 1/p$, with $p, q \in (1, \infty)$. Let $E^s_{p,q}$ mean either $B^s_{p,q}$ or $F^s_{p,q}$. If $s \geq 0$, then $E^s_{p,q}(\Omega)$ is the restriction of $E^s_{p,q}(\mathbb{R}^n)$ to Ω; if $s < 0$, then $E^s_{p,q}(\Omega)$ is the dual of the closure of $C^\infty_0(\Omega)$ in $E^{-s}_{p',q'}(\Omega)$, with $1/p + 1/p' = 1/q + 1/q' = 1$. Put

$$E^s_{p,q}(A, \Omega) := \bigl\{ u \in E^s_{p,q}(\Omega) : Au \in E^{-1+1/p}_{p,q}(\Omega) \bigr\},$$
$$\|u, E^s_{p,q}(A, \Omega)\| := \|u, E^s_{p,q}(\Omega)\| + \|Au, E^{-1+1/p}_{p,q}(\Omega)\|.$$

The space $E^s_{p,q}(A, \Omega)$ is Banach, and $C^\infty(\overline{\Omega})$ is dense in it.

Theorem 1. *Let $s \leq 2l - 1 + 1/p$. Then the mapping*

$$\Lambda : (u, v_1, \ldots, v_\lambda) \mapsto (f, g_1, \ldots, g_{l+\lambda}), \quad \textit{where } u \in C^\infty(\overline{\Omega}) \textit{ and } v_1, \ldots, v_\lambda \in C^\infty(\Gamma), \tag{1}$$

extends uniquely (by continuity) to bounded linear operators

$$\Lambda : B^s_{p,q}(A, \Omega) \oplus \bigoplus\nolimits_{k=1}^{\lambda} B^{s+r_k-1/p}_{p,q}(\Gamma) \to B^{-1+1/p}_{p,q}(\Omega) \oplus \bigoplus\nolimits_{j=1}^{l+\lambda} B^{s-m_j-1/p}_{p,q}(\Gamma),$$
$$\Lambda : F^s_{p,q}(A, \Omega) \oplus \bigoplus\nolimits_{k=1}^{\lambda} B^{s+r_k-1/p}_{p,p}(\Gamma) \to F^{-1+1/p}_{p,q}(\Omega) \oplus \bigoplus\nolimits_{j=1}^{l+\lambda} B^{s-m_j-1/p}_{p,p}(\Gamma). \tag{2}$$

These operators are Fredholm with kernel N and index α. The range of each of these operators consists of all vectors $(f, g_1, \ldots, g_{l+\lambda})$ that belong to the target space and satisfy

$$(f, w)_\Omega + \sum\nolimits_{j=1}^{l+\lambda} (g_j, h_j)_\Gamma = 0 \quad \textit{for all} \quad (w, h_1, \ldots, h_{l+\lambda}) \in N^+. \tag{3}$$

Here, $(\cdot,\cdot)_\Omega$ and $(\cdot,\cdot)_\Gamma$ are extensions of the inner products in $L_2(\Omega)$ and $L_2(\Gamma)$ by continuity.

A similar result holds true for the Nikolskii space $B^s_{p,\infty}$ [1].

Given $0 < \varrho \in C^\infty(\Omega)$, we introduce the following weighted Banach spaces:

$$\varrho E^s_{p,q}(\Omega) := \{\varrho w : w \in E^s_{p,q}(\Omega)\}, \quad \|f, \varrho E^s_{p,q}(\Omega)\| := \|\varrho^{-1} f, E^s_{p,q}(\Omega)\|,$$
$$E^s_{p,q}(A, \varrho, \Omega) := \{u \in E^s_{p,q}(\Omega) : Au \in \varrho E^{s-2l}_{p,q}(\Omega)\},$$
$$\|u, E^s_{p,q}(A, \varrho, \Omega)\| := \|u, E^s_{p,q}(\Omega)\| + \|Au, \varrho E^{s-2l}_{p,q}(\Omega)\|.$$

Let ∂_ν denote the differentiation operator along the inner normal to the boundary of Ω.

Theorem 2. *Let $s < 2l - 1 + 1/p$. Suppose that a positive function $\varrho \in C^\infty(\Omega)$ is a (pointwise) multiplier on $B^{2l-s}_{p',q'}(\Omega)$ or $F^{2l-s}_{p',q'}(\Omega)$ and satisfies*

$$\partial^j_\nu \varrho = 0 \ \text{ on } \Gamma \ \text{ whenever } j \in \mathbb{Z} \ \text{ and } \ 0 \leqslant j < 2l - s - 1 + 1/p. \tag{4}$$

Then mapping (1) where $Au \in \varrho B^{s-2l}_{p,q}(\Omega)$ or $Au \in \varrho F^{s-2l}_{p,q}(\Omega)$ extends uniquely (by continuity) to bounded linear operators

$$\Lambda : B^s_{p,q}(A, \varrho, \Omega) \oplus \bigoplus_{k=1}^{\lambda} B^{s+r_k-1/p}_{p,q}(\Gamma) \to \varrho B^{s-2l}_{p,q}(\Omega) \oplus \bigoplus_{j=1}^{l+\lambda} B^{s-m_j-1/p}_{p,q}(\Gamma),$$
$$\Lambda : F^s_{p,q}(A, \varrho, \Omega) \oplus \bigoplus_{k=1}^{\lambda} B^{s+r_k-1/p}_{p,p}(\Gamma) \to \varrho F^{s-2l}_{p,q}(\Omega) \oplus \bigoplus_{j=1}^{l+\lambda} B^{s-m_j-1/p}_{p,p}(\Gamma), \tag{5}$$

resp. These operators are Fredholm with kernel N and index α. The range of each of these operators consists of all vectors $(f, g_1, \dots, g_{l+\lambda})$ that belong to the target space and satisfy (3).

The following result gives an example of the above weight function ϱ.

Theorem 3. *Let $s < 2l-1+1/p$, and let a positive function $\varrho_1 \in C^\infty(\Omega)$ equal the distance to Γ in a neighbourhood of Γ. Assume that $\delta \geq 2l-s-1+1/p \in \mathbb{Z}$ or that $\delta > 2l-s-1+1/p \notin \mathbb{Z}$. Then the function $\varrho := \varrho_1^\delta$ satisfies the hypotheses of Theorem 2.*

In (2) and (5), the spaces over Γ are independent of q in contrast to the spaces over Ω. This suggests that the set of all $u \in F^s_{p,q}(\Omega)$ such that Au satisfies a relevant condition does not depend on q. The following two theorems give such conditions. Let $p, q, r \in (1, \infty)$.

Theorem 4. *Let $s \leq 2l - 1 + 1/p$, and suppose that a Banach space Q is continuously embedded in $F^{-1+1/p}_{p,\min\{q,r\}}(\Omega)$. Then*

$$\{u \in F^s_{p,q}(\Omega) : Au \in Q\} = \{u \in F^s_{p,r}(\Omega) : Au \in Q\}, \tag{6}$$
$$\|u, F^s_{p,q}(\Omega)\| + \|Au, Q\| \asymp \|u, F^s_{p,r}(\Omega)\| + \|Au, Q\|. \tag{7}$$

As usual, $\asymp$ means equivalence of norms.

Theorem 5. *Let $s < 2l-1+1/p$. Suppose that a positive function $\varrho \in C^\infty(\Omega)$ is a multiplier on the space $F^{2l-s}_{p',q'}(\Omega)$ and satisfies (4). Suppose also that a positive function $\mu \in C^\infty(\Omega)$ is a multiplier on $F^{2l-s}_{p',r'}(\Omega)$, where $1/r + 1/r' = 1$, and satisfies condition (4) in which ϱ is replaced with μ. Let a Banach space Q be continuously embedded in $\varrho F^{s-2l}_{p,q}(\Omega)$ and $\mu F^{s-2l}_{p,r}(\Omega)$. Then (6) and (7) hold true.*

These results were obtained together with A. A. Murach in [2].

1. Murach A. A., Chepurukhina I. S., Elliptic problems with rough boundary data in Nikolskiy spaces. Reports of NAS of Ukraine (*Ukrainian*), 2021, No. 3, 3–10.
2. Murach A. A., Chepurukhina I. S., Elliptic problems in Besov and Sobolev–Triebel–Lizorkin spaces of low regularity. Reports of NAS of Ukraine, 2021, No. 6, 3–11.